Priestley's Representation Theorem

Austin Mohr

April 21, 2010

1 Preliminaries

Definition 1.1. *An ideal I of a lattice is a join-closed downset. An ideal is said to be prime provided that, whenever $a \wedge b \in I$, either $a \in I$ or $b \in I$.*

Theorem 1.2. *(Prime Ideal Theorem) Let $\boldsymbol{L}$ be a distributive lattice and I be an ideal of $\boldsymbol{L}$. For any element $a \in L$ with $a \notin I$, there exists a prime ideal containing I and missing a.*

Theorem 1.3. *(Compactness Theorem) Let $\boldsymbol{L}$ be a distributive lattice. Given subsets S and T of L with the property that every prime ideal containing S hits T, there exist finite subsets S_0 and T_0 of S and T, respectively, such that every prime ideal containing S_0 hits T_0.*

Theorem 1.4. *(Alexander's Subbase Theorem) A topological space is compact if and only if every subbasic open cover of the space admits a finite subcover.*

Lemma 1.5. *Given a collection $\{A_\alpha \mid \alpha \in \Gamma\}$ of clopen subsets of a compact space, there exists a finite subcollection $\{A_i \mid i \leq n\}$ such that $\bigcap_{\alpha \in \Gamma} A_\alpha = \bigcap_{i=1}^{n} A_i$.*

2 Priestley's Representation Theorem

Definition 2.1. *Let Y be a set partially ordered by $\leq$ with least element 0 and greatest element 1, and let τ be a compact topology on Y. The system $\langle Y, \leq, 0, 1, \tau \rangle$ is called a Priestley space provided that, for all $x, y \in Y$, there is a clopen downset containing y but not x whenever $x \nleq y$.*

Theorem 2.2. *(Priestley's Representation Theorem)*
Let $\boldsymbol{L}$ be a distributive lattice and $\mathcal{P}$ be the collection of prime ideals of $\boldsymbol{L}$ ordered by inclusion. (Hence, $\mathcal{P}$ has least element $\emptyset$ and greatest element L.) Let τ be the topology generated by the subasis

$$\{e(a) \mid a \in L\} \cup \{e(a)^c \mid a \in L\},$$

where $e(a)$ denotes the set of elements of $\mathcal{P}$ missing a.

1. *$\langle \mathcal{P}, \subseteq, \emptyset, \boldsymbol{L}, \tau \rangle$ is a Priestley space.*
2. *$\{e(a) \mid a \in L\}$ is exactly the set of proper, nonempty, clopen downsets of $\mathcal{P}$.*

Proof. Evidently, $\mathcal{P}$ is partially ordered by $\subseteq$ with least element $\emptyset$ and greatest element L. To verify that $\langle \mathcal{P}, \subseteq, \emptyset, \mathbf{L}, \tau \rangle$ is a Priestley space, it remains to show that τ is compact and that, for all $I, J \in \mathcal{P}$ with $I \not\subseteq J$, there is a clopen downset containing J but not I.

To show that τ is compact, let $\mathscr{C}$ be a subbasic open cover of $\mathcal{P}$ (by Alexander's Subbase Theorem, the subbasic open covers are sufficient to demonstrate compactness). Define the sets $S = \{a \mid e(a) \in \mathscr{C}\}$ and $T = \{a \mid e(a)^c \in \mathscr{C}\}$. Note that both S and T are nonempty, since $\emptyset \notin e(a)^c$ for any $a \in L$ and $L \notin e(a)$ for any $a \in L$. Now, consider any ideal I containing S and suppose, for the purpose of contradiction, that $I \cap T = \emptyset$. By the Prime Ideal Theorem, we can find a prime ideal P with $I \subseteq P$ and $P \cap T = \emptyset$. By construction, $S \subseteq P$, and so $P \notin e(a)$ for any $a \in S$. Similarly, since $P \cap T = \emptyset$, $P \notin e(a)^c$ for any $a \in T$. We conclude that $\mathscr{C}$ does not cover $\mathcal{P}$, which is a contradiction. Hence, it must be that $I \cap T \neq \emptyset$. In other words, every ideal containing S must intersect T. By the Compactness Theorem, we can find finite subsets S_0 and T_0 of S and T, respectively, such that every prime ideal containing S_0 intersects T_0. Let $\mathscr{C}'$ denote the collection $\bigcup_{a \in S_0} e(a) \cup \bigcup_{a \in T_0} e(a)^c$. Evidently, $\mathscr{C}'$ is a finite subcollection of $\mathscr{C}$. We claim that it also covers $\mathcal{P}$, thus establishing the compactness of τ. To that end, let P be a prime ideal. If P contains S_0, then by the previous remarks, P intersects T_0, and so $P \in e(a)^c$ for some $a \in T_0$. If P does not contain S_0, then P misses some element of $a \in S_0$, and so $P \in e(a)$.

For the remaining claim, let $I, J \in \mathcal{P}$ with $I \not\subseteq J$. Choose any element $a \in I$ with $a \notin J$. Evidently, $J \in e(a)$, which is clopen, since $e(a)^c$ is an element of the subbase. To see that $e(a)$ is also a downset, let K be any prime ideal contained in J. Since $a \notin J$, it follows that $a \notin K$, and so $K \in e(a)$, as desired.

We show next that $\{e(a) \mid a \in L\}$ is exactly the set of proper, nonempty, clopen downsets of $\mathcal{P}$. We have just shown that any $e(a)$ is a clopen downset of $\mathcal{P}$. Furthermore, each $e(a)$ is nonempty (as $\emptyset \in e(a)$ for all $a \in L$) and proper (as $L \notin e(a)$ for any $a \in L$). Hence, the set $\{e(a) \mid a \in L\}$ is contained in the set of propery, nonempty, clopen downsets of $\mathcal{P}$.

Before proving the other inclusion, we first claim that the collection $\{e(a) \mid a \in L\}$ is a base for the subspace $\mathcal{P} \setminus \{L\}$. Indeed,

$$\begin{aligned}
P \in \mathcal{P} \setminus \{L\} &\Leftrightarrow P \text{ is a proper prime ideal of } L \\
&\Leftrightarrow \text{there exists } b \in L \text{ with } b \notin P \\
&\Leftrightarrow P \in e(b) \text{ for some } b \in L \\
&\Leftrightarrow P \in \bigcup_{a \in L} e(a),
\end{aligned}$$

and hence $\mathcal{P} \setminus \{L\} = \bigcup_{a \in L} e(a)$. Next, suppose that $P \in e(b) \cap e(c)$ for some $b, c \in L$. By definition, P is an ideal missing both b and c, and hence P misses $b \wedge c$ since P is prime. Furthermore, $e(b \wedge c) \subseteq e(b) \cap e(c)$, since any downset missing $b \wedge c$ must miss both the larger elements b and c. Thus, $P \in e(b \wedge c) \subseteq e(b) \cap e(c)$, as desired.

Let now G be a proper, nonempty, clopen downset of $\mathcal{P}$. For some prime ideal $I \in G$, denote by A_I the set $\bigcap\{e(a) \mid a \notin I\}$. Evidently, $I \in A_I$, as $I \in e(a)$ for all $a \notin I$. Let now J be another prime ideal belonging to A_I. We see that $J \in e(a)$ for all $a \notin I$. That is, J misses every element that I misses (possibly more), and so $J \subset I$. Since G is a downset, it follows that $A_I \subset G$.

We have now that

$$G = \bigcup_{P \in G} A_P,$$

As we are working in a compact space, we have, for all $P \in \mathcal{P}$

$$\begin{aligned}
A_P &= \bigcap\{e(a) \mid a \notin P\} \\
&= \bigcap\{e(a_i) \mid \text{some finite subcollection of the } e(a)\} \quad \text{(by 1.5)} \\
&= e\left(\bigwedge_{i=1}^{n_P} a_i\right) \\
&= e(a_P) \qquad \text{(some } a_P \in L\text{)},
\end{aligned}$$

and so

$$G = \bigcup_{P \in G} e(a_P).$$

Since G is a closed subset of a compact space, G is also compact. Hence, there exists a finite subcollection $\{b_i \mid i \leq n\}$ of the a_P such that

$$G = \bigcup_{i=1}^{n} e(b_i).$$

Finally, we claim that

$$\bigcup_{i=1}^{n} e(b_i) = e\left(\bigvee_{i=1}^{n} b_i\right),$$

thus finishing the proof. For ease of notation, we show the claim for $n = 2$ and reach the desired conclusion via induction. Let now $P \in e(b) \cup e(c)$. By definition, P misses one of b or c. Since P is a downset, it must also miss the larger element $b \vee c$. Next, let $P \in e(b \vee c)$. Since P misses $b \vee c$, it must miss at least one of b or c, as P is join-closed. Hence, $P \in e(b) \cup e(c)$. □