Math 776 Homework

Austin Mohr

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1 Problem 1

Theorem 1.1. *The following assertions are equivalent for a graph T:*

(i) *T is a tree;*

(ii) *Any two vertices of T are linked by a unique path in T;*

(iii) *T is minimally connected, i.e. T is connected but $T - e$ is disconnected for every edge $e \in T$;*

(iv) *T is maximally acyclic, i.e. T contains no cycle but $T + xy$ does, for any two non-adjacent vertices $x, y \in T$.*

Proof. ($i \Rightarrow ii$) Let T be a tree. Suppose, for the purpose of contradiction, that there are distinct vertices x and y in T that are not connected by a unique path in T. If they are not connected at all, then T is disconnected, which is a contradiction with the fact that T is a tree. Assume, then, that there are distinct xy-paths P_1 and P_2. Denote by z the vertex of least distance from x such that $z \neq x$ and z belongs to both P_1 and P_2. Observe that such a vertex is guaranteed to exist, as $y \neq x$ and y belongs to both P_1 and P_2. Now, xP_1zP_2x is a cycle in T, which is a contradiction with the fact that T is a tree. Therefore, any two vertices in T are linked by a unique path in T.

($ii \Rightarrow iii$) Let T be a graph where any two vertices are linked by a unique path in T. Clearly, T is connected (to reach a vertex from any other, simply follow the unique path in T). To see that T is in fact minimally connected, choose any edge $e \in T$. Denote the endpoints of e by x and y. As the edge e constitutes a path between x and y, it must be that e is the unique path between these two vertices. Hence, that there is no xy-path in $T - e$ (i.e. the removal of e disconnects T). Therefore, T is minimally connected.

($iii \Rightarrow i$) Let T be minimally connected. Suppose, for the purpose of contradiction, that T is not a tree. As T is connected, it must be that T possesses a cycle. The presence of a cycle guarantees at least two paths between the components of G that it separates (if indeed the cycle separates G into more than one component). Hence, $T - e$ is connected for any e in the cycle, which is a contradiction with the fact that T is minimally connected. Therefore, T is a tree.

($i \Rightarrow iv$) Let T be a tree. By definition, T is acyclic. To see that T is in fact maximally acyclic, let x and y be distinct non-adjacent vertices in T and let P denote an xy-path in T

(which must exist, as T is connected). It follows that $xPyx$ is a cycle in $T + xy$. Therefore, T is maximally acyclic.

($iv \Rightarrow i$) Let T be maximally acyclic. Suppose, for the purpose of contradiction, that T is not a tree. As T is acyclic, it must be that T is disconnected. Let x and y be vertices in two disconnected components of T. We see that $T + xy$ is acyclic (otherwise, T would be connected), which is a contradiction with the fact that T is maximally acyclic. Therefore, T is a tree. □

2 Problem 2

Definition 2.1. For graphs G and H, G is called uniquely H-saturated if it does not contain H as a subgraph, but $G+e$ contains a single (unlabeled) copy of H for each $e \in \binom{V(G)}{2} \setminus E(G)$.

In the following, let G be a uniquely K_3-saturated graph.

Lemma 2.2. *G is connected.*

Proof. K_3 has no cut-edges, so it cannot be obtained by adding an edge between two distinct components. □

Lemma 2.3. *G has diameter 2.*

Proof. If G has diameter 1, then G is a single edge, and so possesses too few vertices to support a K_3.

Suppose now (for the purpose of contradiction) that G has diameter greater than 2. Let the vertices x and y witness G's diameter. This means that the shortest x-y path has length at least 3, and so introducing the edge xy to G gives a cycle of length at least 4. More importantly, introducing the edge xy does *not* give a K_3, which contradicts the fact that G is uniquely K_3 saturated. □

Lemma 2.4. *G can contain no cycle other than C_5.*

Proof. As C_3 is the same as K_3, G cannot contain a C_3 by the definition of unique K_3-saturation.

Adding an edge between antipodal vertices of a C_4 results in the creation of two K_3's, which violates the definition of *unique* K_3-saturation. Hence, G cannot contain a C_4.

Let now $k \geq 6$. Antipodal vertices (or nearly antipodal vertices) of C_k have distance at least $\frac{k}{2} \geq 3$, and so any graph containing a C_k has diameter at least 3. Hence, by **??**, G cannot contain a C_k. □

Proposition 2.5. *G is either a star or one of a finite list of graphs.*

Proof. We first claim that, if G is a tree, then G is a star. To see this, suppose (for the purpose of contradiction) that G is not a star. Let v belong to the center of G. As G is not a star, there is a vertex u such that $u \notin N(v)$. That is, the unique u-v path in G has distance at least 2. Let w be a neighbor of v not lying on the u-v path (which must exist, as v is central). The unique u-w path in G has distance at least 3, and so the diameter of G is at least 3, which is a contradiction with **??**. Hence, if G is a tree, it is a star.

Suppose now that G contains a cycle. We have demonstrated that G has diameter 2 (by **??**) and girth 5 (by **??**). According to the Hoffman-Singleton Theorem, there are only finitely-many graphs that possess this property. □

3 Problem 3

Definition 3.1. Let $d \in \mathbb{N}$ and $V = \{0,1\}^d$; thus, V is the set of all 0-1 sequences of length d. The graph on V in which two such sequences form an edge if and only if they differ in exactly one position is called the <u>d-dimensional cube</u> (denoted Q_d).

Proposition 3.2. *The average degree of the d-dimensional cube is d. (In fact, the d-dimensional cube is d-regular.)*

Proof. Consider a vertex $v \in \{0,1\}^d$. As v is a binary string of length d, there are precisely d other binary strings (also of length d) that differ from v in exactly one position. By the above definition, v is adjacent to each of the vertices represented by these binary strings, and so v has degree d. Hence, the d-dimensional cube is d-regular, and so has average degree d. □

Proposition 3.3. *The number of edges of the d-dimensional cube is $d \cdot 2^{d-1}$.*

Proof. We make use of the handshaking lemma, i.e. for any graph G

$$|E(G)| = \frac{1}{2} \sum_{v \in V(G)} d(v)$$

By problem 3a, this becomes

$$\begin{aligned} |E(Q_d)| &= \frac{1}{2} \sum_{v \in \{0,1\}^d} d(v) \\ &= \frac{1}{2} \cdot |\{0,1\}^d| \cdot d \\ &= \frac{1}{2} \cdot 2^d \cdot d \\ &= d \cdot 2^{d-1} \end{aligned}$$

□

Proposition 3.4. *The diameter of the d-dimensional cube is d.*

Proof. We first claim that $diam(Q_d) \leq d$. To see this, let $v, w \in \{0,1\}^d$. Define the path $P = v_0 v_1 \ldots v_d$, where $v_0 = v$ and

$$v_{i+1} = \begin{cases} v_i & \text{if } v_i \text{ and } w \text{ agree at position } i+1 \\ u & \text{otherwise, where } u \text{ is the unique vertex that differs from } v_i \text{ only at position } i+1 \end{cases}$$

(We will ignore multiple occurrences of a vertex in this path.) By the definition of Q_d, we are assured that P is connected (i.e. we can always reach the vertex u). Furthermore, P is acylic, as the construction can never return to a previously visited vertex. Finally, the construction gives $v_d = w$. Hence, P is indeed a $v - w$ path. Now, in the worst case, we will add an edge at each step, giving a path of length d. Hence, the distance between v and w is at most d, i.e. $diam(Q_d) \leq d$.

In the above algorithm, let $v = 0^d$ and $w = 1^d$. Since v and w differ at every bit, we will have to add an edge at each step, and so the upper bound is realized. Therefore, $diam(Q_d) = d$. □

Proposition 3.5. *The girth of the d-dimensional cube is 4 for $d \geq 2$.*

Proof. We first claim that the girth of Q_d is at least 4. Suppose, to the contrary, that we can find a cycle of length 3. Let u, v, and w be the (distinct) vertices of this cycle. Since $u \sim v$, u and v differ at exactly one bit (say, bit i). Similarly, $v \sim w$ means that v and w differ at exactly one bit (say, bit j). Now, it cannot be that $i = j$, or else $u = w$. So, u and w differ at two bits, yet $u \sim w$, which is a contradiction. Hence, the girth of Q_d is at least 4.

Now, let x be any binary string of length $d - 2$. Observe that $00x \sim 10x \sim 11x \sim 01x \sim 00x$ is a cycle of length 4 in Q_d. Therefore, the girth of Q_d is exactly 4. □

Proposition 3.6. *The circumference of the d-dimensional cube is 2^d for $d \geq 2$.*

Proof. (by induction on d)
Basis: $Q_2 = C_4$, and so has a cycle of length $4 = 2^2$.
Induction: Suppose Q_k has a cycle of length 2^k. Partition $V(Q_{k+1})$ into sets V_1 and V_2, where

$$V_1 = \{v \in V(G) \mid v \text{ begins with a 0 bit}\}$$
$$V_2 = \{v \in V(G) \mid v \text{ begins with a 1 bit}\}$$

Observe that the induced subgraph on V_1 is isomorphic to Q_k (define the isomorphism to simply delete the first bit of a vertex), and so there is a cycle on length 2^k on V_1. Similarly, we can find a cycle of length 2^k on V_2. Without loss of generality, let it be that the edge between 0^d and 010^{d-2} belongs to the cycle on V_1 (some isomorphic labeling of the vertices of V_1 can ensure this). Similarly, require that the edge between 10^{d-1} and 110^{d-2} belongs to the cycle on V_2. In addition to these two cycles, include the edge between 0^d and 10^{d-1} and the edge between 010^{d-2} and 110^{d-2}. The result is a 4-cycle on these vertices. Hence, the removal of the edge between 0^d and 010^{d-2} and the edge between 10^{d-1} and 110^{d-2} forms a single cycle of length 2^d, as desired. □

4 Problem 4

Proposition 4.1. *Let G be a graph containing a cycle C. If G contains a path of length at least k between two vertices of C, then G contains a cycle of length at least $\sqrt{k}$.*

Proof. Let P be a path of length at least k between two vertices of C, and let $m = |V(P) \cap V(C)|$. Observe that, if $m \geq \sqrt{k}$, then $|V(C)| \geq m \geq \sqrt{k}$. Assume, then, that $m < \sqrt{k}$. Partition P into P_i for $1 \leq i \leq m-1$ where $P = P_1 \cdots P_{m-1}$. As $|P| \geq k$, there is some P_k such that

$$\begin{aligned} |P_k| &\geq \frac{k}{m-1} \\ &> \frac{k}{\sqrt{k}-1} \\ &> \sqrt{k}. \end{aligned}$$

Denote the ends of P_k by u and v. We have that uP_kvCu is a cycle of length at least $\sqrt{k}+1 > \sqrt{k}$, as desired. □

5 Problem 5

Proposition 5.1. *A line graph is claw-free (i.e. it contains no induced $K_{1,3}$'s).*

Proof. Suppose, for the purpose of contradiction, that G is such that $L(G)$ contains a claw. There must exist edges e, f, g, h in G such that

i. e is incident to each of f, g, and h

ii. f, g, and h are pairwise non-incident

Now, e has only two endpoints in G. By the Pigeonhole Principle, it must be that some endpoint of e is incident to more than one of f, g, and h, and so some pair of f, g, and h are incident, which is a contradiction with contention (ii). Therefore, no configuration of G can produce $L(G)$ containing a claw. □

6 Problem 6

Remark 6.1. The following problem is modified slightly to consider only k-regular graphs for $k \geq 2$. The only connected, 1-regular graph is a single edge, and Diestel's definition of k-connectedness insists that $|G| > k$.

Proposition 6.2. *For $k \geq 2$, any k-regular, connected, bipartite graph is 2-connected.*

Proof. Let $k \geq 2$, and let G be a k-regular, connected, bipartite graph on partite sets A and B. Suppose, for the purpose of contradiction, that G possesses a cutvertex v. Without loss of generality, assume $v \in A$. Let C be some component of $G - v$. We count $|E(C)|$ in two ways.

Consider the vertices of $A \cap V(C)$. As $v \in A$, v is not adjacent to any vertex in $A \cap V(C)$. Hence, each vertex in $V(C)$ has degree k in C, and so k divides $|E(C)|$.

On the other hand, consider the vertices of $B \cap V(C)$. As v is a cutvertex for G, v is adjacent to at least one vertex in $B \cap V(C)$. Let m be the number of vertices of $B \cap V(C)$

that are adjacent to v. Observe that, if all k of v's neighbors belonged to $B \cap V(C)$, removing v would not disconnect the graph. Hence, $m < k$. Now, we have that m of the vertices in $B \cap V(C)$ have degree $k-1$ in C and the remaining vertices have degree k in C. It follows that

$$\begin{aligned} |E(C)| &= m(k-1) + (|B \cap V(C)| - m)k \\ &= mk - m + |B \cap V(C)|k - mk \\ &= |B \cap V(C)|k - m \end{aligned}$$

which is not divisible by k, contradicting the previous claim that $|E(C)|$ is divisible by k.

Therefore, G does not possess a cutvertex, and so G is 2-connected. □

7 Problem 7

Definition 7.1. Let G be a connected graph, and let $r \in G$ be a vertex. Starting from r, move along the edges of G, going whenever possible to a vertex not visited so far. If there is no such vertex, go back along the edge by which the current vertex was first reached (unless the current vertex is r; then stop). This procedure is known as depth-first search.

Proposition 7.2. *The edges traversed during a depth-first search form a normal spanning tree in G with root r.*

Proof. Let G be a connected graph and let T be a subgraph of G resulting from the depth-first search procedure beginning at some vertex r in G. Observe that, since G is connected, T spans G. Moreover, since the procedure never visits the same vertex twice (save for the backtracking step, during which no edge is introduced), T is acyclic. Hence, T is a spanning tree of G.

Now, let u and v be a pair of adjacent vertices in G. Without loss of generality, let it be that u was visited first by the depth-first search procedure. To show that T is in fact normal in G, it suffices to show that vertices u and v are comparable in the tree order of T (since T is a spanning tree of G). When the depth-first search procedure reaches the vertex u (either the first time or during one of the subsequent backtracking steps), we will have v adjacent to u in G and v unvisited by the procedure, so the edge between u and v will be added to T. Hence, when the procedure terminates, u will lie on the unique rv-path in T. In other words, $u \preceq v$ in the tree order of T. Therefore, T is a normal spanning tree in G. □

8 Problem 8

Definition 8.1. Let $d(u, v)$ denote the distance between vertices u and v in G. The eccentricity of a vertex u, written $\epsilon(u)$, is $\max\limits_{v \in V(G)} d(u, v)$. The center of a grpah is the subgraph induced by the vertices of minimum eccentricity.

Theorem 8.2. *(Jordan) The center of a tree is a vertex or an edge.*

Proof. We use induction on the number of vertices in a tree T, denoted $n(T)$.

Basis step: If $n(T) \leq 2$, then the center is the entire tree.

Inductive step: Let $n(T) > 2$. Form the subgraph T' by deleting every leaf of T. Clearly, T' is a tree. Since the internal vertices on paths between leaves of T remain, T' has at least one vertex.

Now, every vertex at minimum dinstance in T form a vertex $u \in V(T)$ is a leaf (otherwise, the path reaching it from u can be extended farther). Since all the leaves have been removed and no path between two other vertices uses a leaf, $\epsilon_{T'}(u) = \epsilon_T(u) - 1$ for every $u \in V(T')$. Also, the eccentricity of a leaf in T is greater than the eccentricity of its neighbor in T. Hence, the vertices minimizing $\epsilon_T(u)$ are the same as the vertices minimizing $\epsilon_{T'}(u)$.

We have shown that T and T' have the same center. By the inductive hypothesis, the center of T' is a vertex or an edge. $\square$

Lemma 8.3. *Let $G \cong H$ under the isomorphism ϕ. For all vertices u and v in G, $d(u, v) = d(\phi(u), \phi(v))$.*

Proof. Let u and v be distinct vertices of G. Denote the shortest uv-path by $ux_1 \cdots x_n v$. Since $G \cong H$ under ϕ,

$$\begin{gathered} u \sim x_1 \Leftrightarrow \phi(u) \sim \phi(x_1) \\ \vdots \\ x_n \sim v \Leftrightarrow \phi(x_n) \sim \phi(v). \end{gathered}$$

Hence, the shortest $\phi(u)\phi(v)$-path in H is $\phi(u)\phi(x_1) \cdots \phi(x_n)\phi(v)$, and so $d(u, v) = d(\phi(u), \phi(v))$. $\square$

Proposition 8.4. *Every automorphism of a tree fixes a vertex or an edge.*

Proof. Let T and T' be trees that are isomorphic under ϕ and let U be the set of vertices of minimal eccentricity in T. By **??**, U contains exactly one or two vertices. Define $\phi(U)$ to be the set $\{\phi u \mid u \in U\}$. By **??**, $\phi(U)$ is the set of vertices minimal eccentricity in T'. Now, if U contains a single vertex u, then $u = \phi(u)$ (i.e. ϕ fixes a vertex). If U contains two vertices u and v, then either $u = \phi(u)$ and $v = \phi(v)$ or $u = \phi(v)$ and $v = \phi(u)$. In either case, ϕ fixes the edge uv. Therefore, ϕ fixes either a vertex or an edge, as desired. $\square$

9 Problem 9

Proposition 9.1. *The dimension of the cycle space of a graph $G = (V, E)$ with k components is $|E| - |V| + k$.*

Proof. Recall that, for a connected graph H, $\dim \mathcal{C}(H) = |E(H)| - |V(H)| + 1$. Now, G is the direct union of k connected graphs (its components). Denote the components of G by

$G_i = (V_i, E_i)$ for $1 \leq i \leq k$. We have,

$$\begin{aligned}\dim \mathcal{C}(G) &= \sum_{i=1}^{k} \mathcal{C}(G_i) \\ &= \sum_{i=1}^{k} |E_i| - |V_i| + 1 \\ &= \sum_{i=1}^{k} |E_i| - \sum_{i=1}^{k} |V_i| + \sum_{i=1}^{k} 1 \\ &= |E| - |V| + k.\end{aligned}$$

□

Proposition 9.2. *The dimension of the cut space of a graph with k components is*

Proof. Recall that, for a connected graph H, $\dim \mathcal{C}^*(H) = |V(H)| - 1$. Now, G is the direct union of k connected graphs (its components). Denote the components of G by $G_i = (V_i, E_i)$ for $1 \leq i \leq k$. We have,

$$\begin{aligned}\dim \mathcal{C}^*(G) &= \sum_{i=1}^{k} \mathcal{C}^*(G_i) \\ &= \sum_{i=1}^{k} |V_i| - 1 \\ &= \sum_{i=1}^{k} |V_i| - \sum_{i=1}^{k} 1 \\ &= |V| - k.\end{aligned}$$

□

10 Problem 10

Proposition 10.1. *The only finite, connected graph that is isomorphic to its own line graph is a cycle.*

Proof. Let $G = (V, E)$ be a finite, connected graph. Denote its line graph $L(G)$ by (V', E'). If $G \cong L(G)$, then

$$\begin{aligned}|V| &= |V'| && \text{(true of pair of isomorphic graphs)} \\ &= |E| && \text{(as } L(G) \text{ is a line graph).}\end{aligned}$$

Now, since G is connected, it must possess a spanning tree, which requires $|V| - 1$ of the available $|V|$ vertices. As this spanning tree is maximally acyclic, the remaining edge in G forms a unique cycle C in G.

Suppose now, for the purpose of contradiction, that G is not itself a cycle. There is a vertex u in G that is adjacent to C, and the uniqueness of C implies that u does not belong to some other cycle. Let v be u's neighbor in C and let x and y be v's neighbors in C. The induced subgraph on the vertices u, v, x, and y is a claw, which is a contradiction with **??**. Therefore, G is a cycle. $\square$

11 Problem 1

For edges $e, e' \in G$, write $e \sim e'$ if either $e = e'$ or e and e' lie on some common cycle in G.

Proposition 11.1. *The relation $\sim$ is an equivalence relation on $E(G)$ whose equivalence classes are the edge sets of the non-trivial blocks of G.*

Proof. To show that $\sim$ is an equivalence relation on $E(G)$, we verify that it is reflexive, symmetric, and transitive.

The relation $\sim$ is reflexive.

Proof. By definition, $e \sim e'$ whenever $e = e'$, so $e \sim e$. $\square$

The relation $\sim$ is symmetric.

Proof. Let $e, e' \in E(G)$ with $e \sim e'$. Either $e = e'$ or e and e' lie on the some common cycle in G. In either case, switching the roles of e and e' yields the same statement (either $e' = e$ or e' and e lie on some common cycle in G), and so $e' \sim e$. $\square$

The relation $\sim$ is transitive.

Proof. Let $e, f, g \in E(G)$ be three distinct edges with $e \sim f$ and $f \sim g$. That is, e and f lie on a cycle C_1 and f and g lie on a cycle C_2. Our aim is to produce a cycle in G containing both e and g.
If $C_1 = C_2$, we are done. Otherwise, both C_1 and C_2 contain the edge f, so the collection $C_1 \cup C_2 \setminus \{f\}$ is a cycle containing both e and g. $\square$

Now, let $e \in E(G)$. We know that e belongs to a unique equivalence class A under $\sim$, and that e belongs to a unique block B in G. To show that the equivalence classes of $E(G)$ with respect to $\sim$ are precisely the non-trivial blocks of G, it suffices to show that $A = E(B)$.

($\subseteq$) Let $f \in A$. By definition, $e \sim f$, and so either $e = f$ or e and f lie on some common cycle in G. If $e = f$, then certainly e and f belong to the same block of G, and so $f \in B$. Suppose now that e and f lie on some common cycle C in G. Now, since $e \in B$, the maximal 2-connected subgraph of G that contains C is in fact B. Hence, $f \in B$.

($\supseteq$) Let $f \in B$. Now, B is either a bridge or a maximal 2-connected subgraph of G. In the first case, we have that $|B| = 1$, and so it must be that $e = f$. Hence, $e \sim f$. In the second case, B is 2-connected, and so B can be constructed from some cycle by successively adding H-paths to graphs H already constructed (Dietel 3.1.3). Hence, we can find some cycle in B containing both e and f, and so $e \sim f$. $\square$

12 Problem 2

Proposition 12.1. *Let G be a 3-connected graph, and let xy be an edge of G. The graph G/xy is 3-connected if and only if $G - \{x, y\}$ is 2-connected.*

Proof. ($\Rightarrow$) Let G/xy be 3-connected and suppose, for the purpose of contradiction, that $G - \{x, y\}$ is not 2-connected. That is, $G - \{x, y\}$ is either disconnected or 1-connected.

In the first case, we see that $\{x, y\}$ is a separating set for G, which is a contradiction with the fact that G is 3-connected.

In the second case, there exists a cutvertex z of $G - \{x, y\}$, and so $\{x, y, z\}$ is a separating set for G. Let v be the vertex resulting from the contraction of the edge xy. We see that $\{v, z\}$ is a separating set for G/xy, which is a contradiction with the fact that G/xy is 3-connected.

($\Leftarrow$) Let $G - \{x, y\}$ be 2-connected and suppose, for the purpose of contradiction, that G/xy is not 3-connected. That is, G/xy is either disconnected or is at most 2-connected.

In the first case, it must be that G itself is disconnected, as the contraction of an edge never disconnects a graph. This is contrary to the fact that G is 3-connected.

For the second case, suppose G/xy is 2-connected (the same argument will hold if it is merely 1-connected). Let $\{z_1, z_2\}$ be the set of separating vertices. Now, it must be that one of these vertices resulted from the contraction of the edge xy (otherwise, $\{z_1, z_2\}$ is a separating set of size 2 for G). Without loss of generality, $\{x, y, z_1\}$ is a separating set for G. Hence, $\{z_1\}$ is a separating set for $G - \{x, y\}$, which is a contradiction with the fact that $G - \{x, y\}$ is 2-connected. □

13 Problem 3

Proposition 13.1. *Let k be an integer. Any two partitions of a finite set into k-sets admit a common choice of representatives.*

Proof. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two partitions of the same finite set into k-sets. Define the graph $G = (V, E)$ where

$$\begin{aligned} V &= \mathcal{P}_1 \cup \mathcal{P}_2 \\ uv \in E &\text{ whenever } u \cap v \neq \emptyset. \end{aligned}$$

As $\mathcal{P}_1$ and $\mathcal{P}_2$ are partitions, there can be no edges between vertices corresponding to subsets of the same partition. Hence, G is bipartite with partite sets $\mathcal{P}_1$ and $\mathcal{P}_2$.

Now, the question of determining whether or not these two partitions admit a common choice of representatives is equivalent to establishing that G possesses a matching. By Hall's Theorem, we need only verify that G satisfies $|N(S)| \geq |S|$ for all $S \subseteq \mathcal{P}_1$ (the marriage condition). Observe first that for any $S \subseteq \mathcal{P}_1$, if we view the elements of S as subsets of the partition $\mathcal{P}_1$, we have

$$\bigcup_{A \in S} A \leq \bigcup_{A \in N(S)} A$$

by definition of our adjacency relation (if an element of our finite set is contained in some subset in S, then it must be contained in some subset of $N(S)$). Since all subsets are of size k, the equation above ensures that $|N(S)| \geq |S|$, and so the marriage condition is satisfied. Hence, G possesses a 1-factor. Therefore, any two partitions of a finite set into k-sets admit a common choice of representatives. □

14 Problem 4

Definition 14.1. A graph G is called (vertex-) transitive if, for any two vertices $v, w \in G$, there is an automorphism of G mapping v to w.

Proposition 14.2. *Every transitive connected graph of even order contains a 1-factor.*

Proof. Let M be a matching in G of maximum size. For any graph, there is $S \subseteq V(G)$ such that

i. S is matchable to $\mathcal{C}_{G-S}$;

ii. Every component of $G - S$ is factor-critical.

From the remarks in Diestel, we know that the maximum matching M is obtained by taking the edges guaranteed by the matchability of S to $\mathcal{C}_{G-S}$ as well as the 1-factor of $C - v$ for each $C \in \mathcal{C}_{G-S}$, where v is the vertex already matched to S.

Suppose now, for the purpose of contradiction, that M does not constitute a 1-factor. By the remarks above, it must be that there is some component of $\mathcal{C}_{G-S}$ that is not matched to S. By the transitivity of G, there is an automorphism mapping a vertex in this component into S. In this isomorphic copy of G, every edge of M is maintained. As before, we are guaranteed that S is matchable to $\mathcal{C}_{G-S}$. Taking M together with this new edge constitutes a larger matching, which is a contradiction with the maximality of M. Therefore, it must be that M is indeed a 1-factor. □

15 Problem 5

Proposition 15.1. *A graph G contains k independent edges if and only if $q(G - S) \leq |S| + |G| - 2k$ for all sets $S \subseteq V(G)$.*

Proof. (⇒) (Partial) Let G contain k independent edges. If we consider the vertex set S given to us by Theorem 2.2.3, we have

$$\begin{aligned} k &\leq |M| \\ &= \frac{1}{2}(|G| + |S| - |\mathcal{C}_{G-S}|) \\ 2k &\leq |G| + |S| - |\mathcal{C}_{G-S}| \\ |\mathcal{C}_{G-S}| &\leq |G| + |S| - 2k. \end{aligned}$$

I do not see how to apply this fact to general subsets of $V(G)$.

(⇐) Let G be such that $q(G - S) \leq |S| + |G| - 2k$ for all sets $S \subseteq V(G)$. In particular, this inequality holds for the vertex set S_0 satisfying

i. S_0 is matchable to $\mathcal{C}_{G-S_0}$;

ii. Every component of $G - S_0$ is factor-critical,

which is guaranteed to exist. From the remarks in Diestel, we know that the maximum matching M is obtained by taking the edges guaranteed by the matchability of S_0 to $\mathcal{C}_{G-S_0}$ as well as the 1-factor of $C - v$ for each $C \in \mathcal{C}_{G-S_0}$, where v is the vertex already matched to S_0. It follows that

$$\begin{aligned}|M| &= |S_0| + \frac{1}{2}(|G| - |S_0| - |\mathcal{C}_{G-S_0}|)\\ &\geq |S| + \frac{1}{2}(|G| - |S| - (|S| + |G| - 2k))\\ &= k.\end{aligned}$$

Therefore, G possesses at least k disjoint edges. □

16 Problem 6

Given a graph G, let $\alpha(G)$ denote the largest size of a set of independent vertices in G.

Lemma 16.1. *Let G be a connected, leafless graph. There exists a cycle C of G a vertex $v \in V(C)$ such that $N(v) \subseteq V(C)$.*

Proof. Let $P = v_0 v_1 \cdots v_m$ be a longest path in G. Since G is leafless, it must be that v_m has degree at least 2. As P is of maximal length in G, it must be that the neighbors of v_m all lie on P (otherwise, P adjoined with some neighbor results in a path with length strictly greater than that of P). Let v_k be the neighbor of v_m of smallest index and consider the cycle C defined by $v_k P v_m v_k$. By construction, every neighbor of v_m is some v_l with $l \geq k$, and so every neighbor of v_m lies on C, as desired. □

Proposition 16.2. *The vertices of G can be covered by at most $\alpha(G)$ disjoint subgraphs, each isomorphic to a cycle or a K^2 or a K^1.*

Proof. (by strong induction on $\alpha(G)$)

Base Step

If $\alpha(G) = 1$, then G is the complete graph, and so possesses a Hamiltonian cycle (i.e. a single cycle covering all the vertices of G).

Inductive Step

Suppose $\alpha(G) = k$ that the vertices of any graph G' can be covered by at most $\alpha(G')$ disjoint subgraphs whenever $\alpha(G') < k$. We proceed by finding a subgraph H of G ismorphic to one of K_1, K_2, or a cycle containing a vertex v with $N(v) \subseteq V(H)$. If we can produce such a subgraph, observe that $\alpha(G - H) < \alpha(G)$ (the vertex $v \in H$ can be added to any independent set in $G - H$ to make a strictly larger independent set in G). By the inductive hypothesis, $G - H$ can be covered by at most $\alpha(G) - 1$ disjoint copies of K_1, K_2, or cycles, which implies that G can be covered by at most $\alpha(G)$ disjoint copies of these graphs (since H and $G - H$ are disjoint and H is isomorphic to one of the desired graphs).

It remains to show that such a subgraph H indeed exists for any graph.

Suppose G contains an isolated vertex v. We take $H = \{v\}$. As v has no neighbors, it trivially satisfies the condition that $N(v) \subseteq V(H)$.

Supopse G contains a leaf v. Denote its unique neighbor in G by u. We take $V(H) = \{u, v\}$ and $E(H) = \{uv\}$. It follows that $N(v) = \{u\} \subset V(H)$.

If G contains no isolated vertices nor leaves, then we take H to be the cycle guaranteed by **??**. □

17 Problem 8

Definition 17.1. A split graph $G = (V, E)$ is a graph whose vertex set admits a partition $V = C \cup I$ so that $G[C]$ is complete and $G[I]$ is an independent set.

Proposition 17.2. *Any connected split graph with $I \neq \emptyset$ satisfies $\kappa(G) = \lambda(G) = \delta(G)$.*

Proof. Let G be a connected split graph and let $V(G) = C \cup I$ as in the definition. Consider a vertex v of minimum degree in G.

If $v \in C$, then $\deg v = |C|$. Furthermore, each vertex in I has degree equal to $|C|$ (as $\deg v$ is minimum), and so must be adjacent to every vertex in C. Hence, in order to disconnect G, we must remove either the $|C|$ edges incident to a single vertex or we must remove all $|C|$ vertices from C. Thus, in the case where $v \in C$, $\kappa(G) = \lambda(G) = \delta(G)$.

If $v \in I$, then $\deg v \leq |C|$. Furthermore, the component of $G - C$ containing v is of size 1. Hence, v can only be separated from G by removing the $\deg v$ edges incident to v or the $\deg v$ vertices in C that are adjacent to v. Now, as $G[C]$ is complete, we know that $\kappa(G[C]) = \lambda(G[C]) = |C|$, which is at least as big as $\deg v$. Hence, the edge cutset and vertex cutset described above are in fact minimal. Thus, in the case where $v \in I$, $\kappa(G) = \lambda(G) = \delta(G)$. □

18 Problem 9

Definition 18.1. The matching number $\nu(G)$ of a graph G is the size of a maximum matching in G.

Proposition 18.2. *A connected graph G is factor-critical if and only if $\nu(G) = \nu(G - u)$ for every $u \in V(G)$.*

Proof. ($\Rightarrow$) Let G be factor-critical. For any $u \in V(G)$, $G - u$ contains a 1-factor, and so $\nu(G-u) = \frac{1}{2}(|V(G)|-1)$. As G is of odd degree (all factor-critical graphs are of odd degree), G itself can contain no 1-factor. Hence, the number of vertices in a maximum matching of G can be at most $|V(G)| - 1$, but this is obtained by the matching demonstrated in $G - u$ for any $u \in V(G)$. Hence, $\nu(G) = \frac{1}{2}(|V(G)|-1)$. Therefore, $\nu(G) = \nu(G-u)$ for any $u \in V(G)$.

($\Leftarrow$) (Partial) Let G be such that $\nu(G) = \nu(G - u)$ for every $u \in V(G)$. Let M be a matching in G of maximum size. For any graph, there is $S \subseteq V(G)$ such that

i. S is matchable to $\mathcal{C}_{G-S}$;

ii. Every component of $G - S$ is factor-critical.

From the remarks in Diestel, we know that the maximum matching M is obtained by taking the edges guaranteed by the matchability of S to $\mathcal{C}_{G-S}$ as well as the 1-factor of $C - v$ for each $C \in \mathcal{C}_{G-S}$, where v is the vertex already matched to S. This gives

$$|M| = |S| + \frac{1}{2}(|G| - |S| - |\mathcal{C}_{G-S}|).$$

Now, let $u \in S$ and consider a maximum matching M' of $G - u$. By hypothesis, it must be that $|M| = |M'|$.

My problem here similar to that in problem 5. I can say something about the particular vertex set S given by Theorem 2.2.3, but I cannot seem to generalize to an arbitrary subset of $V(G)$. □

19 Problem 10

Definition 19.1. We will say that a graph possesses property T if it is a finite, connected graph in which every block is a triangle and every vertex is of degree 2 or 4.

Lemma 19.2. *Let G_1 and G_2 be factor critical. The graph obtained by adding an edge between any vertex in G_1 and any vertex in G_2 contains a 1-factor.*

Proof. Let G be constructed as above. Let $u \in G_1$ and $v \in G_2$ be the endvertices of the edge between G_1 and G_2. As G_1 is factor-critical, $G_1 - u$ contains a 1-factor M_1. Similarly, $G_2 - v$ contains a 1-factor M_2. Now, u is unmatched in M_1 and v is unmatched in M_2, so the set $M_1 \cup M_2 \cup \{uv\}$ is a 1-factor of G. □

Proposition 19.3. *If G possesses property T, then G is factor critical.*

Proof. (by strong induction on $|G|$)

Base Step

The smallest graph with property T is K_3. Observe that the removal of any vertex from K_3 leaves a single edge remaining which is a 1-factor its indicent vertices. Hence, K_3 is factor-critical.

Inductive Step

Let G possess property T. To see that G is factor-critical, consider the removal of any vertex v of G.

If v is of degree 2, then its removal does not disconnect the graph (v belongs to a single K_3, which is 2-connected). Furthermore, v belongs to a triangular block, and so its removal leaves only a single edge remaining in the block. Hence, the graph resulting from the removal of v is a pair of graphs G_1 and G_2 connected by a bridge. Now, each of G_1 and G_2 either possesses property T or is a single vertex. If either of G_1 or G_2 is a single vertex, we will relax our defintion slightly to allow them to be called factor-critical. Otherwise, we invoke the inductive hypothesis (as $|G_1|$ and $|G_2|$ are both strictly less than $|G|$) to conclude that they are factor-critical. By Lemma **??**, $G - v$ contains a 1-factor.

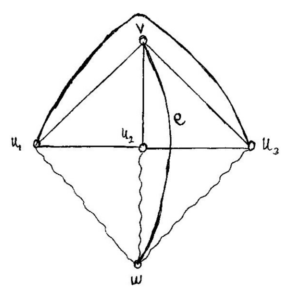
If v is of degree 4, then its removal is equivalent to the removal of two overlapping triangular blocks. Hence, this removal may disconnect the graph into two components. Treating each component separately, we can view each as having subgraphs G_1 and G_2 connected by a bridge with G_1 and G_2 both possessing property T or being a single vertex. By the argument above, each component of $G - v$ contains a 1-factor.

Therefore, for any $v \in G$, $G - v$ contains a 1-factor. That is, G is factor-critical. □

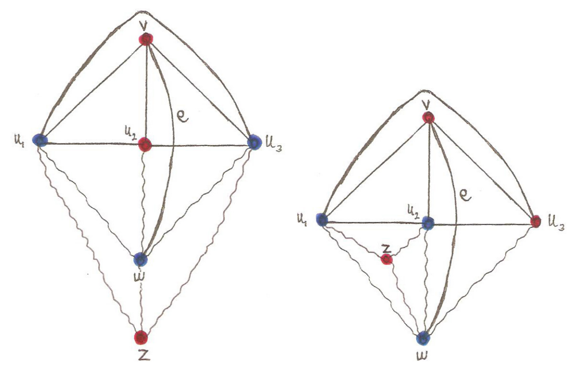
20 Problem 2

Proposition 20.1. *Adding a new edge to a maximal planar graph of order at least 6 always produces both a* TK^5 *and a* $TK_{3,3}$ *subgraph.*

Proof. Let G be a maximal planar graph of order at least 6. We have immediately that G is 3-connected. Before adding the new edge e to G, consider the vertices v and w which are to be its endpoints. As G is 3-connected, there are 3 disjoint vw-paths. Since v is not adjacent to w (else we could add no edge between them), we can find vertices u_1, u_2, and u_3 lying one on each path that are neighbors of v but not of w. By the edge-maximality of G, u_1, u_2, and u_3 induce a cycle. Finally, adding e gives the TK^5 (see figure - solid lines represent true edges, while wiggly lines represent topological edges).



To show the existence of a $TK_{3,3}$, repeat the same construction up to the point where we add e. Since G has order at least 6, there is another vertex z distinct from those previously mentioned. The construction allows for two cases: either z lies outside the region bounded by the topological cycle vu_1wu_2v or it lies inside one of the faces of this region (they are all equivalent). In either case, the edge-maximality of G gives the existence of new topological edges e_1, e_2, and e_3 that ensure a $TK_{3,3}$ (see figures).



□

21 Problem 3

Definition 21.1. A graph is called outerplanar if it has a drawing in which every vertex lies on the boundary of the outer face.

Proposition 21.2. *A graph is outerplanar if and only if it contains neither K^4 nor $K_{2,3}$ as a minor.*

Proof. (⇒) Let G be outerplanar. Define the graph G' by placing a new vertex v in the unbounded face of G and connecting v to every vertex of G. Since G is outerplanar, these edges can be added without inducing a crossing, and so G' is planar. By Kuratowski's Theorem, G' contains no K^5 nor $K_{3,3}$ minor. Hence, it must be that G contains no K^4 nor $K_{2,3}$ minor (else adding the vertex v and the corresponding edges would result in a K^5 or $K_{3,3}$ minor in G').

(⇐) Let G contain no K^4 nor $K_{2,3}$ minor. Construct the graph G' as before. Evidently, G' contains no K^5 nor $K_{3,3}$ (else G would contain a K^4 or $K_{2,3}$ minor), and so G' is planar. A priori, we do not know where v is located in a planar drawing of G'. To address this, project G' to the sphere S^2. Using an appropriate homeomorphism from S^2 to S^2 and then projecting back to the plane, we can obtain a planar drawing of G' in which v lies in the unbounded face of G. The fact that v can reach every vertex of G without inducing a crossing implies that every vertex of G is on the boundary of the unbounded face. That is, G is outerplanar. □

22 Problem 4

Proposition 22.1. *A 2-connected plane graph is bipartite if and only if every face is bounded by an even cycle.*

Proof. (⇐) In a 2-connected plane graph, every face is bounded by a cycle. As the graph is bipartite, it contains no odd cycle, and so every face must be bounded by an even cycle.

(⇒) In a 2-connected plane graph G, the facial cycles generate the entire cycle space. Hence, C can be written as the sum of even facial cycles. Now, any edge lies on exactly two faces. Furthermore, an edge in the sum is canceled (i.e. does not appear in C) if and only if it appears in exactly two of the facial cycles generating G. Hence, the cancellation of edges does not change the parity of the number of edges in the sum, and so we conclude that C is even. Therefore, every cycle of G is even, and so G is bipartite. □

23 Problem 5

Proposition 23.1. *The set of critically 3-chromatic graphs is precisely the set of odd cycles.*

Proof. ($\supseteq$) Let G be an odd cycle on $2n+1$ vertices. We have immediately that G is non-bipartite, and so $\chi(G) \geq 3$. Observe that, for all $v \in V(G)$, $G - v$ is a path, and so is bipartite. In other words, $G-v$ is 2-chromatic. Hence, G is 3-chromatic (take any 2-coloring of $G-v$ and use a third color for v), and so critically 3-chromatic, as v was chosen arbitrarily.

($\subseteq$) Let G be critically 3-chromatic. Since G is 3-chromatic, it is non-bipartite, and so contains an odd cycle. Consider the cycle C in G of smallest size. Observe that there is no chord of C in G, as this would yeild a smaller odd cycle.

Suppose now, for the purpose of contradiction, that C is a proper subgraph of G. It follows that there is some vertex $v \in V(G)$ that does not lie on C. Hence, $C \subseteq G - v$. Since $G - v$ contains an odd cycle, it is non-bipartite, and so $\chi(G - v) = 3$, which is contrary to the fact that G is critically 3-chromatic. Therefore, it must be that $G = C$. That is, G is an odd cycle. □

24 Problem 6

Proposition 24.1. *Given $k \in \mathbb{N}$, there is a constant $c_k > 0$ such that every large enough graph G with $\alpha(G) \leq k$ contains a cycle of length at least $c_k|G|$.*

Proof. We know that $|G| \leq \alpha(G)\chi(G)$. It follows that

$$\begin{aligned} \frac{|G|}{\alpha(G)} &\leq \chi(G) \\ &\leq \max\{\delta(H) \mid H \subseteq G\} + 1. \end{aligned}$$

Let H_0 be the subgraph witnessing $\max\{\delta(H) \mid H \subseteq G\}$ (that is, $\max\{\delta(H) \mid H \subseteq G\} = \delta(H_0)$).

Now, if $\delta(H_0) \geq 2$, we are guaranteed to find a cycle of length at least $\delta(H_0) + 1$ in H_0 (and so also in G). Hence, we can take c_k to be $\frac{1}{k}$ to obtain

$$\begin{aligned} \frac{|G|}{k} &\leq \frac{|G|}{\alpha(G)} && \text{(since } \alpha(G) \leq k) \\ &\leq \delta(H_0) + 1, \end{aligned}$$

as desired.

It remains to show that, given $k \geq 1$, every sufficiently large graph with $\alpha(G) \leq k$ contains a subgraph with minimum degree at least 2. Arguing by contrapositive, suppose that every subgraph of G has minimum degree at most 1. It must be that G is acyclic, and so G is a forest. As every forest is bipartite, we have that $\alpha(G) \geq \left\lceil \frac{|G|}{2} \right\rceil$ (one of the partite sets accounts for at least half of the vertices of G). By making G arbitrarily large, we can force $\alpha(G) > k$ for any k. Having established the contrapositive, we conclude that every sufficiently large graph with $\alpha(G) \leq k$ contains a subgraph with minimum degree at least 2, thus completing the proof. □

25 Problem 7

Proposition 25.1. *For every $k \in \mathbb{N}$, there is a triangle-free k-chromatic graph.*

Proof. (by strong induction on k) For $|G| = 1$, the claim holds trivially. Assume inductively that there are triangle-free graphs $G_1, \ldots, G_{k-1}$ with $\chi(G_i) = i$ for $1 \leq i \leq k-1$. Let W denote the vertex set $V(G_1) \times \cdots \times V(G_{k-1})$ (the ordinary Cartesian product of the sets $V(G_i)$). Now, construct the graph $G = (V, E)$ with

$$V = V(G_1) \cup \cdots \cup V(G_{k-1}) \cup W$$
$$E = \{xy \mid xy \in G_i \text{ for some } i \text{ or } x \in W \text{ and } y = x_i \text{ for some } i\}$$

where x_i denotes the i^{th} coordinate of x as a tuple. Observe that there are no edges between vertices of W nor between graphs G_i and G_j for $i \neq j$. We claim that G_k is triangle-free and k-chromatic, thus establishing the proposition.

Evidently, G_k is triangle-free, since, for $1 \leq i \leq k-1$, G_i is triangle-free and vertices of W have exactly one neighbor in each of the (disjoint) G_i.

It is clear that G_k is k-colorable, since we can inductively color the vertices of the G_i for $1 \leq i \leq k-1$ using at most $k-1$ colors. Since no edge lies between vertices of W, we can color all of W using a new color k. To see that fewer colors will not suffice, choose, for $1 \leq i \leq k-1$, a vertex $v_i \in V(G_i)$ such that v_i is colored differently than v_j for all $j < i$. Thus, $k-1$ distinct colors are represented in the collection of v_i. Now, the vertex $(v_1, \ldots, v_{k-1}) \in W$ is adjacent to each of the v_i, and so must be colored differently than each of them. Hence, G_k requires at least k colors, and so is k-chromatic. □

26 Problem 8

Let $G = (V, E)$ and $H = (V', E')$ be graphs.

Definition 26.1. The Cartesian/direct product $G \Box H$ is the graph on $V \times V'$ with

$$(a, b) \sim (c, d) \Leftrightarrow ((a = c) \wedge (b \sim d)) \vee ((b = d) \wedge (a \sim c)).$$

Definition 26.2. The categorical/tensor product $G \times H$ is the graph on $V \times V'$ with

$$(a, b) \sim (c, d) \Leftrightarrow (a \sim c) \wedge (b \sim d).$$

Definition 26.3. The strong/normal product GH is the graph on $V \times V'$ with

$$(a, b) \sim (c, d) \Leftrightarrow ((a = c) \vee (a \sim c)) \wedge ((b = d) \vee (b \sim d)).$$

Definition 26.4. The lexicographic/replacement product $G[H]$ is the graph on $V \times V'$ with

$$(a, b) \sim (c, d) \Leftrightarrow (a \sim c) \vee ((a = c) \wedge (b \sim d)).$$

Proposition 26.5. $\chi(G \square H) = \max\{\chi(G), \chi(H)\}$

Proof. Let $k = \max\{\chi(G), \chi(H)\}$. Observe first that $G \square H$ contains a copy of G (fix $b = d$ in $V(H)$ and let a and c run through $V(G)$). Similarly, $G \square H$ contains a copy of H. Hence, $\chi(G \square H) \geq k$. It remains to show that $G \square H$ is indeed k-colorable. To that end, let $g : V(G) \to \{0, \ldots, \chi(G) - 1\}$ define a proper coloring of G. Similarly, let $h : V(H) \to \{0, \ldots, \chi(H) - 1\}$ define a proper coloring of H. Define the coloring

$$\begin{aligned} &f : V(G) \times V(H) \to \{0, \ldots, k - 1\} \\ &f(u, v) = g(u) + h(v) \text{ (mod k)}. \end{aligned}$$

Evidently, f makes use of k colors. To see that f indeed defines a proper k-coloring on $G \square H$, let $(u, v) \sim (u', v')$ in $G \square H$. The definition of adjacency in the Cartesian product allows for two cases. In one case, $u = u'$ and $v \sim v'$, which implies that $g(u) = g(u')$ and $h(v) \neq h(v')$. In the other case, $v = v'$ and $u \sim u'$, which implies that $h(v) = h(v')$ and $g(u) \neq g(u')$. In either case, we see that $g(u) + h(v) \neq g(u') + h(v')$. That is, $f(u, v) \neq f(u', v')$, and so f is indeed a proper k-coloring of $G \square H$. □

Proposition 26.6. $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$

Proof. The symmetry of the tensor product allows us to assume, without loss of generality, that $\chi(G) \leq \chi(H)$. Let g define a proper $\chi(G)$-coloring on G. Now, define the coloring f on $G \times H$ by $f(u, v) = g(u)$. Evidently, f uses $\chi(G)$ colors. To see that f indeed defines a proper coloring on $G \times H$, let $(u, v) \sim (u', v')$ in $G \times H$. It follows that

$$\begin{aligned} (u, v) \sim (u', v') &\Rightarrow u \sim u' \\ &\Rightarrow g(u) \neq g(u') \\ &\Rightarrow f(u, v) \neq f(u', v'). \end{aligned}$$

Hence, $\chi(G \times H) \leq \chi(G)$. That is, $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$. □

Proposition 26.7. $\max\{\chi(G), \chi(H)\} \leq \chi(GH) \leq \chi(G[H]) \leq \chi(G)\chi(H)$

Proof. For the first inequality, observe that GH contains a copy of G (fix $b = d$ in $V(H)$ and let a and c run through $V(G)$). Similarly, GH contains a copy of H. Hence, $\chi(GH) \geq \max\{\chi(G), \chi(H)\}$.

For the second inequality, observe that $GH \subseteq G[H]$ since

$$\begin{aligned} & ((a = c) \vee (a \sim c)) \wedge ((b = d) \vee (b \sim d)) \\ \Leftrightarrow \quad & ((a = c) \wedge ((b = d) \vee (b \sim d))) \vee ((a \sim c) \wedge ((b = d) \vee (b \sim d))) \\ \Rightarrow \quad & ((a = c) \wedge ((b = d) \vee (b \sim d))) \vee (a \sim c)) \\ \Leftrightarrow \quad & ((a = c) \wedge (b = d)) \vee ((a = c) \wedge (b \sim d)) \vee (a \sim c) \\ \Leftrightarrow \quad & ((a = c) \wedge (b \sim d)) \vee (a \sim c) \qquad \text{(since loops are not allowed).} \end{aligned}$$

Hence, $\chi(GH) \leq \chi(G[H])$.

For the third inequality, let g define a $\chi(G)$-coloring on G and h define a $\chi(H)$-coloring on H. Define the coloring f on $G[H]$ by $f(u,v) = (g(u), h(v))$. Evidently, f uses $\chi(G)\chi(H)$ colors. To see that f is indeed a proper coloring of $G[H]$, let $(u,v) \sim (u',v')$ in $G[H]$. It follows that

$$\begin{aligned}(u,v) \sim (u',v') &\Leftrightarrow (u \sim u') \vee ((u = u') \wedge (v \sim v')) \\ &\Rightarrow (u \sim u') \vee (v \sim v') \\ &\Rightarrow (g(u) \neq g(u')) \vee (h(v) \neq h(v')) \\ &\Leftrightarrow (g(u), h(v)) \neq (g(u'), h(v')) \\ &\Leftrightarrow f(u,v) \neq f(u',v').\end{aligned}$$

Hence, $\chi(G[H]) \leq \chi(G)\chi(H)$. □

27 Problem 9

Definition 27.1. The <u>thickness</u> $\Theta(G)$ of a graph G is the minimum k so that $G = \bigcup_{i=1}^{k} G_i$ for some planar graphs G_i, $1 \leq i \leq k$.

Proposition 27.2. *Any graph G with $||G|| > 0$ satisfies*

$$\left\lceil \frac{\delta(G)}{6} \right\rceil \leq \Theta(G) \leq \left\lceil \frac{\Delta G}{2} \right\rceil.$$

Proof. To establish the lower bound, observe that for some i, G_i accounts for a fraction of at least $\frac{1}{\Theta(G)}$ of the edges of G. It follows that,

$$\begin{aligned}\Theta(G)||G_i|| &\geq ||G|| \\ &\geq \frac{1}{2}|G|\delta(G) && \text{(by the Handshaking Lemma)} \\ &\geq \frac{1}{2}|G_i|\delta(G).\end{aligned}$$

Hence, $\Theta(G) \geq \frac{|G_i|\delta(G)}{2||G_i||}$. In fact, $\Theta(G) \geq \left\lceil \frac{|G_i|\delta(G)}{2||G_i||} \right\rceil$, since $\Theta(G)$ must be an integer. Now, as any plane graph on n vertices has at most $3n-6$ edges, we have that $||G_i|| \leq 3|G_i| - 6$, and so

$$\begin{aligned}\Theta(G) &\geq \left\lceil \frac{|G_i|\delta(G)}{2||G_i||} \right\rceil \\ &\geq \left\lceil \frac{|G_i|\delta(G)}{2(3|G_i|-6)} \right\rceil \\ &\geq \left\lceil \frac{|G_i|\delta(G)}{6|G_i|} \right\rceil \\ &\geq \left\lceil \frac{\delta(G)}{6} \right\rceil,\end{aligned}$$

as desired.

To establish the upper bound, suppose that G is regular of even degree. A result of Petersen shows that G possesses a 2-factor. As a 2-factor is planar, we can take this 2-factor to be G_1 (i.e. the first graph contributing toward the thickness of G). Next, remove the edges of this 2-factor from G. By definition of 2-factor, this removal decreases the degree of every vertex of G by exactly 2. Hence, G minus these edges is still regular of even degree, and so possesses another 2-factor which we will take to be G_2. After $\frac{\Delta(G)}{2}$ iterations, the remaining graph will be regular of degree 0. Hence, we can represent G as the union of $\frac{\Delta(G)}{2}$ (planar) 2-factors, and so $\Theta(G) = \frac{\Delta(G)}{2}$ when G is regular of even degree.

It remains to show that, given any graph G, we can extend it to a regular graph of even degree without lowering the thickness. To begin, take a second copy of G (call it G') and add an edge between corresponding vertices of G and G' if the common degree of these vertices is less than $2\left\lceil\frac{\Delta(G)}{2}\right\rceil$. Note that the addition of these vertices and edges can only raise the thickness (if a set of planar graphs can cover this larger graph, then they can surely cover the original G). Call this newly constructed graph H. Now, if H is still not regular of degee $2\left\lceil\frac{\Delta(G)}{2}\right\rceil$, take another copy of H and add edges as before. Continuing in this way, we construct a regular graph of degree $2\left\lceil\frac{\Delta(G)}{2}\right\rceil$ (call it $\tilde{G}$). Combining our previous observations, we have that

$$\begin{aligned}\Theta(G) &\leq \Theta(\tilde{G}) \\ &= \frac{1}{2}\cdot 2\left\lceil\frac{\Delta(G)}{2}\right\rceil \\ &= \left\lceil\frac{\Delta(G)}{2}\right\rceil,\end{aligned}$$

as desired. □

28 Problem 10

Definition 28.1. A chordal graph is a graph with the property that any cycle longer than a C_3 induces a chord.

Definition 28.2. A near-triangulation is a connected graph with a planar embedding so that all inner faces are triangles.

Proposition 28.3. *A finite graph is chordal and outerplanar if and only if it is a disjoint union of K^4-minor free near-triangulations.*

Proof. (⇒) Let G be chordal and outerplanar. Since G is outerplanar, we immediately have that G is K^4-minor free by **??**. Since G is chordal, we see that G cannot contain a facial cycle larger than C_3 (anything larger induces a chord). Hence, every inner face of G is a triangle. That is, G is near-triangular.

(⇐) Let G be a finite K^4-minor free near-triangulation. We first claim that G contains no $K_{2,3}$ minor. Suppose, for the purpose of contradiction, that this is not the case. A planar drawing of $K_{2,3}$ contains two facial cycles of size 4. Since G is near-triangular, it must be that these faces are subdivided into triangles in its $K_{2,3}$ minor. This subdivision yields a TK^4, which implies that G contains a K^4 minor - a contradiction. Now, as G contains no K^4 nor $K_{2,3}$ minor, we have that G is outerplanar by **??**.

We show next that, if G is outerplanar and near-triangular, then it is chordal. To that end, let C be any cycle in G larger than C_3. Since G is near-triangular, the region bounded by C must be subdivided into triangles. Since G is outerplanar, there can be no vertex on the interior of this region, and so the subdivision is accomplished using chords. Hence, G is chordal. □

29 Problem 1

Definition 29.1. The family $\mathcal{C}$ of *cographs* consists of (1) K^1, (2) the complement of any cograph, (3) the disjoint union of any two cographs.

Lemma 29.2. *A cograph can be reduced to a set of isolated vertices by iterated complementation of its components.*

Proof. Let C be a component of a cograph G. If $\overline{C}$ is connected, then both C and $\overline{C}$ are atomic cographs in the sense that they are not the complement of the disjoint union of smaller cographs. The only graph that does not arise in this way is the given graph K^1, and so $C = \overline{C} = K^1$.

It follows that, if $C \neq K^1$, then $\overline{C}$ is disconnected. Hence, we can proceed inductively by taking the complement of the components of $\overline{C}$ until we are left with only copies of K^1. □

Lemma 29.3. *The graph G contains an induced P^3 if and only if $\overline{G}$ does.*

Proof. Let G contain an induced P^3. There are vertices v_1, v_2, v_3, v_4 in G with $v_1 \sim v_2$, $v_2 \sim v_3$, $v_3 \sim v_4$, and all other pairs nonadjacent. Hence, in $\overline{G}$ we have $v_2 \sim v_4$, $v_4 \sim v_1$, $v_1 \sim v_3$, and all other pairs nonadjacent. That is, $\overline{G}$ contains an induced P^3.

The reverse implication follows from the fact that $\overline{\overline{G}} = G$. □

Lemma 29.4. *The disjoint union of perfect graphs is perfect.*

Proof. Let G and H be disjoint perfect graphs. We have immediately that

$$\begin{aligned}
\chi(G \cup H) &= \max\{\chi(G), \chi(H)\} && \text{(since } G \text{ and } H \text{ are disjoint)}\\
&= \max\{\omega(G), \omega(H)\} && \text{(since } G \text{ and } H \text{ are perfect)}\\
&= \omega(G \cup H) && \text{(since } G \text{ and } H \text{ are disjoint).}
\end{aligned}$$

□

Lemma 29.5. *The complement of a perfect graph is perfect.*

Proof. (partial) Let G be a perfect graph. A result of Lovász gives that, for all induced subgraphs H of G, $|H| \leq \alpha(H)\omega(H)$. Now, for any induced subgraph $\overline{H_0}$ of $\overline{G}$, we have

$$\begin{aligned} |\overline{H_0}| &= |H_0| \\ &\leq \alpha(H_0)\omega(H_0) && \text{(since } G \text{ is perfect)} \\ &= \omega(\overline{H_0})\alpha(\overline{H_0}). \end{aligned}$$

Hence, $\overline{G}$ is perfect. □

Proposition 29.6. *The cographs are precisely those graphs which contain no induced* P^3.

Proof. Let $\mathcal{P}$ denote the family of graphs containing no induced P^3. We show that $\mathcal{C} = \mathcal{P}$.

($\subseteq$) Let $G \in \mathcal{C}$. By **??**, G can be reduced to a set of isolated vertices by iterated complementation of its compnents. Combining this fact with **??**, it follows that G contains no induced P^3.

($\supseteq$) (I attempted to emulate the induction proof we sketched yesterday, but every time I tried to force G to have an induced P_3, I seemed to find a way to avoid it without achieving the desired conclusion.) □

Proposition 29.7. *Cographs are perfect.*

Proof. (by strong induction on $|G|$)

If $|G| = 1$, then $G = K^1$, which is perfect.

Suppose now that $|G| = n$ and all cographs on at most $n-1$ vertices are perfect. If G arises as the disjoint union of two smaller (and so perfect) cographs, then G is perfect by **??**. Otherwise, it must be that $\overline{G}$ is the disjoint union of two smaller cographs G_1 and G_2 by **??**. By the inductive hypothesis, each of G_1 and G_2 is perfect, and so their union is perfect by **??**. Finally, **??** gives that G itself is perfect. □

30 Problem 2

Lemma 30.1. *The chromatic polynomial of the cycle on n vertices is* $(\lambda-1)^n + (-1)^n(\lambda-1)$.

Proof. (by induction on n)

For $n = 3$, observe that C_3 is the same as K^3, giving $\lambda(\lambda-1)(\lambda-2) = \lambda^3 - 3\lambda^2 + 2\lambda$ colorings. Comparing this with the proposed formula, we see

$$\begin{aligned} p_{C_3}(\lambda) &= (\lambda-1)^3 + (-1)^3(\lambda-1) \\ &= (\lambda^3 - 3\lambda^2 + 3\lambda - 1) - (\lambda - 1) \\ &= \lambda^3 - 3\lambda^2 + 2\lambda, \end{aligned}$$

and so the formula hold in the case of $n = 3$.

For $n \geq 4$, observe that

$$\begin{aligned}
p_{C_n}(\lambda) &= p_{C_n - e}(\lambda) - p_{C_n/e}(\lambda) && \text{(some } e \in E(C_n)) \\
&= p_{P_{n-1}}(\lambda) - p_{C_{n-1}}(\lambda) \\
&= \lambda(\lambda-1)^{n-1} - \left[(\lambda-1)^{n-1} + (-1)^{n-1}(\lambda-1)\right] \\
&= \lambda(\lambda-1)^{n-1} - (\lambda-1)^{n-1} - (-1)^{n-1}(\lambda-1) \\
&= \lambda(\lambda-1)^{n-1} - (\lambda-1)^{n-1} + (-1)^{n}(\lambda-1) \\
&= (\lambda-1)^{n-1}(\lambda-1) + (-1)^{n}(\lambda-1) \\
&= (\lambda-1)^{n} + (-1)^{n}(\lambda-1),
\end{aligned}$$

as desired. □

Proposition 30.2. *The chromatic polynomial of the wheel graph* $K^1 * C_n$ *is* $\lambda\left[(\lambda-2)^{n-1} + (-1)^{n-1}(\lambda-2)\right]$.

Proof. Let v be the central vertex of the wheel graph W_n. First, choose any of the available λ colors to color v. Since v is adjacent to all other vertices, the chosen color cannot be used again. Hence, it remains to color $W_n - v$ (i.e. C_n) using $\lambda - 1$ colors. Making use of **??**, we have

$$\begin{aligned}
p_{W_n}(\lambda) &= \lambda p_{C_n}(\lambda - 1) \\
&= \lambda\left[(\lambda-2)^{n} + (-1)^{n}(\lambda-2)\right],
\end{aligned}$$

as desired. □

Proposition 30.3. *The chromatic polynomial of* $K_{2,s}$ *is* $\lambda(\lambda-1)\left[(\lambda-1)^{s-1} + (\lambda-2)^{s}\right]$.

Proof. Let v and w be the vertices in the partite set of size 2. We consider two cases.

If v and w are colored the same, then we choose a single color of the available λ colors. As both v and w are adjacent to all of the remaining s vertices, we color using the remaining $\lambda-1$ colors. As these s vertices are independent, we can color freely, and so arrive at $\lambda(\lambda-1)^s$ colorings for this case.

If v and w are colored differently, then there are $\lambda(\lambda-1)$ ways to color them. As before, we use the remaining $\lambda-2$ colors to color the remaining s vertices freely, giving $\lambda(\lambda-1)(\lambda-2)^s$ colorings for this case.

Taken together, we have that

$$\begin{aligned}
p_{K_{2,s}}(\lambda) &= \lambda(\lambda-1)^{s} + \lambda(\lambda-1)(\lambda-2)^{s} \\
&= \lambda(\lambda-1)\left[(\lambda-1)^{s-1} + (\lambda-2)^{s}\right],
\end{aligned}$$

as desired. □

31 Problem 3

Definition 31.1. An *interval* representation of a graph $G = (V, W)$ is a family $\mathcal{I}$ of intervals of the real line and a bijection $\phi : V \to \mathcal{I}$ so that $vw \in E$ if and only if $\phi(v) \cap \phi(w) \neq \emptyset$. Such a representation is "proper" if no element of $\mathcal{I}$ contains another.

Proposition 31.2. *An interval graph is claw-free (no induced $K_{1,3}$) if and only if it has a proper representation.*

Proof. ($\Rightarrow$) We proceeed by establishing the contrapositive. To that end, let G be an interval graph having no proper representation. For all representations, there are intervals I_1 and I_2 such that $I_1 \subseteq I_2$. Furthermore, there are intervals I_3 and I_4 with I_1 properly between them (and so $I_3 \cap I_4 = \emptyset$) and such that $I_2 \cap I_3 \neq \emptyset$ but $I_2 \cap I_1 = \emptyset$ and $I_4 \cap I_3 \neq \emptyset$ but $I_4 \cap I_1 = \emptyset$. Were this not the case, then I_1 could be freely extended to the left or the right so as not to be contained in I_2, thus giving a proper representation. Now, let v_j be the vertex in G corresponding to I_j. We have that v_2 is adjacent to v_1, v_3, and v_4, while the latter three vertices are themselves pairwise nonadjacent. Hence, G is not claw-free.

($\Leftarrow$) We proceed by establishing the contrapositive. To that end, let G be an interval graph that is not claw-free. There are vertices v_1, v_2, v_3, and v_4 such that, without loss of generality, v_1 is adjacent to all of v_2, v_3, and v_4, while the latter three vertices are themselves pairwise nonadjacent. Letting I_j be the interval corresponding to the vertex v_j shows that I_1 intersects nonemptily with each of I_2, I_3, and I_4, while the latter three intervals are themselves pairwise disjoint. This happens only if one of I_2, I_3, or I_4 is contained in I_1. Hence, G has no proper representation. □

32 Problem 4

Proposition 32.1. *The complement of a bipartite graph is perfect.*

Proof. Let G be such that $\overline{G}$ is bipartite. It suffices to show that G is perfect, as any induced subgraph of G is also the complement of a bipartite graph (specifically, the complement of some induced subgraph of $\overline{G}$).

Observe first that $\chi(G) \geq \omega(G)$. Now, the color classes of any $\chi(G)$-coloring contain at most two vertices (else $\overline{G}$ contains a clique of size larger than 2, and so is not bipartite). Let k_1 denote the number of color classes containing a single vertex (and V_1 denote the set of these vertices) and k_2 denote the number color classes containing two vertices. Evidently, those vertices that belong to a color class of size 1 form a clique of size k_1 in G (and so form an independent set of size k_1 in $\overline{G}$). Next, observe that by choosing a single vertex from each color class of size 2, we construct a minimum vertex cover of $\overline{G}$, and so the maximum size of a matching in $\overline{G}$ is k_2 by König's Theorem. We can deduce that, for every edge uv in a maximum matching of $\overline{G}$, one of u or v is incident to V_1. Suppose this is not the case. That is, $u \sim w_1$ and $v \sim w_2$ for some $w_1, w_2 \in V_1$. Since $\overline{G}$ is bipartite, $w_1 \neq w_2$. Hence, we can construct a strictly larger matching by discarding the edge uv and adding the edges uw_1 and vw_2 - a contradiction. Thus, we have that $\overline{G}$ contains an independent set of size at least $k_1 + k_2 = \chi(G)$, so $\alpha(\overline{G}) \geq \chi(G)$. That is, $\omega(G) \geq \chi(G)$. Therefore, $\chi(G) = \omega(G)$ (i.e. G is perfect). □

33 Problem 5

Lemma 33.1. *Every chordal graph has a simplicial vertex.*

Proof. We know that every chordal graph can be constructed recursively by pasting along complete subgraphs starting with complete graphs. In a complete graph, every vertex is simplicial. Now, let G be a chordal graph obtained by pasting two chordal graphs G_1 and G_2 as described above. By the inductive hypothesis, each of G_1 and G_2 has a simplicial vertex. As $G_1 \subset G$, it follows that G also has a simplicial vertex. □

Lemma 33.2. *If v is a simplicial vertex in a graph G whose neighborhood is a clique of size k, then $p_G(\lambda) = (\lambda - k)p_{G-v}(\lambda)$.*

Proof. There are $p_{G-v}(\lambda)$ ways to color $G-v$ using λ colors. Since v is adjacent to a k-clique, we are left with $\lambda - k$ ways to color v. □

Proposition 33.3. *The chromatic polynomial of a chordal graph has only integer roots.*

Proof. Let G be a chordal graph. By **??**, G has a simplicial vertex v. Let v be adjacent to a clique of size k in G. By **??**, we have that $p_G(\lambda) = (\lambda - r)p_{G-v}(\lambda)$. Now, as $G-v$ is chordal, it possesses a simplicial vertex. Proceeding inductively, we can factor p_G into linear factors having integer roots. □

34 Problem 6

Proposition 34.1. *Given two vertex colorings f and g of G with $D = \operatorname{col}(G) + 1$ colors, there exists a sequence of vertex D-colorings $f = c_1, c_2, \ldots, c_{k-1}, c_k = g$ so that c_j differs from c_{j+1} at exactly one vertex for each $j = 1, \ldots, k-1$.*

Proof. (by induction on $|G|$)

If $|G| = 1$, then $\operatorname{col}(G) = 1$, and so there are only two possible colorings of G on $D = 2$ colors. As G has no edges, we can switch from one coloring to another by simply recoloring the single vertex in G.

Let now $|G| = n$ and let v be a vertex of G of least degree. By the inductive hypothesis, there is a sequence of colorings of $G-v$ having the desired properties. Given a coloring of G, proceed to recolor it using the same sequence. This will satisfactorily recolor $G-v$ except in the case where a neighbor of v is assigned the same color as v. As the neighborhood of v is guaranteed to use no more than $\operatorname{col}(G)$ colors at any given stage of the recoloring, it follows that we can interrupt the recoloring of $G-v$, recolor v with an unused color, and proceed with the recoloring of $G-v$. Once $G-v$ is colored appropriately, we finish by assigning the correct color to v which, by the same argument, can always be done. □

35 Problem 7

Proposition 35.1. *Suppose G and H are nonempty and connected. $G \square H$ is planar if and only if either (a) one of G or H is K^1 and the other is planar, (b) one of G or H is K^2 and the other is outerplanar, or (c) one of G or H is P^r for some $r \geq 2$ and the other is either P^s for some $s \geq 2$ or a cycle.*

Proof. (⇒) We first make a couple of observations which will allow us to characterize which G and H give planar $G \square H$.

First, observe that one of G or H must be a tree. If not, then both G and H contain a C_3 minor. Denote the vertices of the C_3 minor in G by u_1, u_2, and u_3 and denote the vertices of the C_3 minor in H by v_1, v_2, and v_3. In $C_3 \square C_3$, we have the $K_{3,3}$ minor with partite sets $\{(u_1, v_1), (u_2, v_2), (u_3, v_3)\}$ and $\{\{(u_1, v_2), (u_1, v_3)\}, \{(u_2, v_1), (u_2, v_3)\}, \{(u_3, v_1), (u_3, v_2)\}\}$. Hence, $G \square H$ is not planar.

Next, observe that if one of G or H has maximum degree 2, then the other has maximum degree at most 2. Suppose this is not the case. Let G have maximum degree 3 and H have maximum degree 2. We have that G contains a $K_{1,3}$ subgraph and H contains a P^2 subgraph. Denote the vertices of the $K_{1,3}$ in G by u_1, u_2, u_3, and u_4, with u_1 being the central vertex. Denote the vertices of the P^2 in H by v_1, v_2, and v_3. Now, in $G \square H$, we have the $K_{3,3}$ minor with partite sets $\{(u_1, v_1), (u_1, v_2), (u_1, v_3)\}$ and $\{\{(u_2, v_1), (u_2, v_2), (u_2, v_3)\}, \{(u_3, v_1), (u_3, v_2), (u_3, v_3)\}, \{(u_4, v_1), (u_4, v_2), (u_4, v_3)\}\}$. Hence, $G \square H$ is not planar.

Now, let H be a tree with maximum degree at most 2. We consider all cases.

If H has maximum degree 0, then $H = K^1$. Evidently, $G \square H = G$. Hence, $G \square H$ is planar if and only if G is planar.

If H has maximum degree 1, then $H = K^2$. Hence, $G \square H$ can be viewed as two copies of G in which an additional edge is added between corresponding vertices. Now, if G is not outerplanar, then every drawing has some vertex which is not on the frontier of the unbounded face in the plane. Hence, this vertex in one copy cannot be connected to its corresponding vertex in the other copy without inducing a crossing. Hence, G must be outerplanar.

If H has maximum degree 2 and is a tree, then $H = P^r$ for some $r \geq 2$. Since G must have maximum degree at most 2, it follows that G can be P^s for some $s \geq 2$ or a cycle. Indeed, both of these possibilities result in planar $G \square H$ (see below).

(⇐) We show that each of the listed choices for G and H indeed results in planar $G \square H$.

If G is planar and H is K^1, then $G \square H \simeq G$, which is planar.

If G is outerplanar and H is K^2, then $G \square H$ can be viewed as 2 copies of G with edges added between corresponding vertices in the copies. Project one copy of G to the sphere. Since G is outerplanar, it can be drawn so that all its vertices lie on some disk and, furthermore, all its edges lie within this disk. Now, an appropriate homeomophism can redraw G so that all its edges lie *outside* this disk. Projecting back to the plane, we can nest the original copy of G inside this "inverted" copy of G and add edges between corresponding vertices without inducing a crossing. Hence, $G \square H$ is planar.

If G is P^s and H is P^r for some $r, s \geq 2$, then $G \square H$ is an $r \times s$ grid, which is planar.

If G is P^s for some $s \geq 2$ and H is a cycle, then $G \square H$ can be realized as s nested copies of H. In addition, an edge is introduced between corresponding vertices of a cycle and the one nested immediately inside it. By this construction, we see that $G \square H$ is planar. □

36 Problem 8

Definition 36.1. A graph G is called χ-unique if $p_G = p_H$ implies $G \simeq H$.

Proposition 36.2. *$K_{n,n}$ is χ-unique.*

Proof. Let G be such that $p_G = p_{K_{n,n}}$. We have that

$$\begin{aligned} |G| &= \deg p_G \\ &= \deg p_{K_{n,n}} \\ &= |K_{n,n}| \\ &= 2n. \end{aligned}$$

Next, observe that

$$\begin{aligned} K_{n,n} \text{ is bipartite} &\Rightarrow p_{K_{n,n}}(2) \neq 0 \\ &\Rightarrow p_G(2) \neq 0 \\ &\Rightarrow G \text{ is bipartite.} \end{aligned}$$

Finally, by Whitney's Broken Circuit Theorem,

$$\begin{aligned} ||G|| &= [x^{n-1}]p_G \\ &= [x^{n-1}]p_{K_{n,n}} \\ &= n^2. \end{aligned}$$

Now, let G have partite sets of size k and $2n-k$. It follows that $||G|| = k(2n-k)$, which equals n^2 only when $k = n$. Hence, G is bipartite with partite sets both of size n. Since we previously determined that $||G|| = n^2$, it follows that $G \simeq K_{n,n}$. $\square$