Math 738N Homework

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April 20, 2012

Problem 4^*

Proposition 1. *Every closed interval of* ω_1 *(endowed with the order topology) is compact.*

Proof. Let $[\alpha, \beta]$ be a closed interval of ω_1. Without loss of generality, assume it is covered by a collection $\mathcal{U}$ of basic open sets of the form $(\xi, \eta]$. We construct a finite subcover $\mathcal{U}'$.

There is a basic open set $(\xi_1, \xi_1']$ belonging to $\mathcal{U}$ that contains β. We include $(\xi_1, \xi_1']$ in the finite subcover $\mathcal{U}'$.

It remains now to cover $[\alpha, \xi_1]$. As before, there is a basic open set $(\xi_2, \xi_2']$ belonging to $\mathcal{U}$ that contains ξ_1. We include $(\xi_2, \xi_2']$ in the finite subcover $\mathcal{U}'$.

Continuing in this way, we produce a descending sequence of ξ_i's. Since this is a descending sequence in a well-ordered set, we know that the sequence is finite. Thus, $(\xi_n, \xi_n']$ contains the left endpoint α for some finite n. Taking $\mathcal{U}'$ to be the set $\{(\xi_i, \xi_i'] \mid 1 \leq i \leq n\}$, we see that $\mathcal{U}'$ is a finite subcover of $[\alpha, \beta]$. □

Proposition 2. *The space* ω_1 *(endowed with the order topology) is sequentially compact.*

Proof. Let $(\xi_n)_{n=1}^{\infty}$ be a sequence in ω_1.

If the sequence contains only finitely-many distinct elements of ω_1, then some element of the sequence appears infinitely-often, which gives a convergent subsequence.

Assume now that the sequence contains infinitely-many distinct elements of ω_1. We may choose indices $n_1, n_2, \ldots$ such that the subsequence $(\xi_{n_i})_{i=1}^{\infty}$

is strictly increasing. Since ω_1 is well-ordered and uncountable, a least upper bound ξ for this subsequence exists.

We claim that $(\xi_{n_i})_{i=1}^{\infty}$ converges to ξ. Indeed, were there a basic open set $(\alpha, \beta]$ containing ξ but missing the sequence, it would contain an ordinal strictly smaller than ξ that is an upper bound for the sequence, contradicting the fact that ξ is the least upper bound for the sequence. □

Problem 5

Theorem 3. *The intersection of countably-many clubs is a club.*

Corollary 4. *If S is a stationary subset of ω_1 and $S = \bigcup_{n=1}^{\infty} S_n$, then at least one S_n is stationary.*

Proof. We prove the contrapositive. For each n, let C_n be a club witnessing the fact that S_n is nonstationary. By the theorem above, $\bigcap_{n=1}^{\infty} C_n$ is a club missing S, so S is nonstationary. □

Problem 6*

Define the *diagonal intersection* of a collection $\{C_\alpha \mid \alpha \in \omega_1\}$ to be the set

$$\Delta \bigcap \{C_\alpha \mid \alpha \in \omega_1\} = \{\theta \mid \theta \in C_\gamma \forall \gamma < \theta\}.$$

Theorem 5. *The diagonal intersection of an ω_1-sequence of clubs is a club.*

Proof. Let D denote the diagonal intersection of $\{C_\alpha \mid \alpha \in \omega_1\}$, where each C_α is a club.

We show first D is closed. To that end, let $\kappa \notin D$. We seek an open set containing κ that is disjoint from D. Since $\kappa \notin D$, there is an ordinal $\alpha_0 < \kappa$ such that $\kappa \notin C_{\alpha_0}$. Since C_{α_0} is closed, there is an open set U containing κ and disjoint from C_{α_0}.

Let $U' = U \cap (\alpha_0, \kappa]$, which is open, contains κ, and is disjoint from C_{α_0}. Now, for all $\theta \in U'$, we have $\alpha_0 < \theta$ and $\theta \notin C_{\alpha_0}$, and so $\theta \notin D$. Thus, U' is an open set containing κ that is disjoint from D, as desired.

We show next D is unbounded. To do so, let κ be any countable ordinal, and put κ into correspondence with ω for the purposes of indexing the sequence to come. We form a sequence $(\xi_{i,j})$ where, for each fixed i, $\xi_{i,j} \in C_i$

for all j. Beginning with $\xi_{1,1} > \kappa$, the sequence is ordered as follows:

$$\begin{aligned}
&\xi_{1,1} < \\
&\xi_{1,2} < \xi_{2,2} < \\
&\xi_{1,3} < \xi_{2,3} < \xi_{3,3} < \\
&\vdots
\end{aligned}$$

Such a sequence exists since each C_i is unbounded. For any fixed i, $\lim_{j\to\infty} \xi_{i,j}$ is equal to the limit of the sequence under the lexicographic order shown above (call this limit ξ). Moreover, $\xi \in C_i$ for all i, since each is closed. That is, $\xi \in \bigcap_{\alpha<\kappa} C_\alpha$. □

Problem 9

A *scattered* space is one in which every nonempty subspace contains an isolated point in its relative topology.

Given a space X, let $X[\alpha]$ denote the collection of isolated points of $X \setminus \bigcup\{X[\beta] \mid \beta < \alpha\}$ (with $X[0]$ denoting the collection of isolated points of X). The α^{th} *Cantor-Bendixson derivative* of a space X is $X \setminus \bigcup\{X[\beta] \mid \beta < \alpha\}$ and is denoted $X^{(\alpha)}$.

Proposition 6. *A space is scattered if and only if $X^{(\alpha)} = \emptyset$ for some α.*

Proof. Let X be scattered. As each $X^{(\alpha)}$ defines a subspace of X, the collection $X[\alpha]$ is nonempty whenever $X^{(\alpha)}$ is. Thus, for any α,

$$\left|\bigcup\{X[\beta] : \beta < \alpha\}\right| > \alpha,$$

and so $X^{(|X|^+)} = \emptyset$ (though a much smaller α may suffice).

For the other direction, let S be a subspace of X. Let also α_S be least such that $S \cap X[\alpha_S] \neq \emptyset$ (which exists since there is α such that $X^{(\alpha)} = \emptyset$). By definition, any point belonging to $S \cap X[\alpha_S]$ is isolated in $X^{(\alpha_S)}$, and so is isolated in the relative topology on S. □

Problems 16 and 17

Let $X = \prod_{\alpha\in\Gamma} X_\alpha$ be a product space. For any subset I of Γ, define

$$\prod_I^* V_\alpha = \bigcap\{\pi_\alpha^\leftarrow V_\alpha \mid \alpha \in I\},$$

where each V_α is a subset of X_α.

Proposition 7. *If $I \cap J = \emptyset$ and $\prod_I^* V_\alpha$ and $\prod_J^* V_\beta$ are nonempty subsets of X, then $\prod_{I\cup J}^* V_\alpha \neq \emptyset$.*

Proof. Without loss of generality, view an element of X as a tuple of length $|I \cup J|$, where the first $|I|$ coordinates correspond to X_α for $\alpha \in I$ and the remaining $|J|$ coordinates correspond to X_β for $\beta \in J$. (If X contains more factors X_γ that are not witnessed by I or J, then these coordinates can be any element of X_γ.)

An element of $\prod_I^* V_\alpha$ is a tuple having any element of $\pi_\alpha^\leftarrow V_\alpha$ in the α^{th} coordinate ($\alpha \in I$) and any element of X_β for the β^{th} coordinate ($\beta \in J$). Similarly, an element of $\prod_J^* V_\beta$ is a tuple having any element of $\pi_\beta^\leftarrow V_\beta$ in the β^{th} coordinate ($\beta \in J$) and any element of X_α for the α^{th} coordinate ($\alpha \in I$).

Since I and J are disjoint, we can take the first $|I|$ coordinates of any element of $\prod_I^* V_\alpha$ and concatenate with the last $|J|$ coordinates of any element of $\prod_J^* V_\beta$ to produce an element of $\prod_{I\cup J}^* V_\alpha$. □

Proposition 8. *If $\mathcal{J}$ is a family of disjoint subsets of Γ, $\prod_J^* V_\alpha \neq \emptyset$ for all $J \in \mathcal{J}$, and $I = \bigcup\{J \mid J \in \mathcal{J}\}$, then $\prod_I^* V_\alpha \neq \emptyset$.*

Proof. Without loss of generality, view an element of X as a tuple of length $|\bigcup \mathcal{J}|$, where each $\alpha \in J \in \mathcal{J}$ corresponds to a unique (by disjointness of the $J \in \mathcal{J}$) factor X_α. (If X contains more factors X_γ that are not witnessed by any $J \in \mathcal{J}$, then these coordinates can be any element of X_γ.)

For each $J \in \mathcal{J}$, an element of $\prod_J^* V_\alpha$ is a tuple having any element of $\pi_\alpha^\leftarrow V_\alpha$ in the α^{th} coordinate ($\alpha \in J$) and any element of X_β for the β^{th} coordinate ($\beta \notin J$).

Since $\mathcal{J}$ is a disjoint family (and so no two sets of $\mathcal{J}$ index the same coordinate), we can form an element of $\prod_{J\in\mathcal{J}}^* V_\alpha$ by choosing an element of $\prod_J^* V_\alpha$ for the α^{th} coordinate for each $J \in \mathcal{J}$ and each $\alpha \in J$. □

Proposition 9. *Let $I \cap J \neq \emptyset$ and $\prod_I^* U_\alpha$ and $\prod_J^* V_\alpha$ be nonempty subsets of X. Their intersection is nonempty if and only if $U_\alpha \cap V_\alpha \neq \emptyset$ for all $\alpha \in I \cap J$.*

Proof. As before, we assume without loss of generality that elements of X are tuples of length $|I \cup J|$.

By the description of the star-product in the previous proof, the β^{th} coordinate of $\prod_I^* U_\alpha \cap \prod_J^* V_\alpha$ is

- an element of U_β if $\beta \in I \setminus J$,
- an element of V_β if $\beta \in J \setminus I$, or
- an element of $U_\beta \cap V_\beta$ if $\beta \in I \cap J$.

If $U_\alpha \cap V_\alpha \neq \emptyset$ for all $\alpha \in I \cap J$, then each of these nonempty, so $\prod_I^* U_\alpha \cap \prod_J^* V_\alpha$ is nonempty.

Conversely, if there exists $\alpha \in I \cap J$ such that $U_\alpha \cap V_\alpha = \emptyset$, then there is no element possible for the α^{th} coordinate, and so $\prod_I^* U_\alpha \cap \prod_J^* V_\alpha$ is empty. $\square$

Problems 23 and 24

Martin's Axiom (MA): If $(P, \leq)$ is an $\uparrow$-ccc poset and $\mathcal{D}$ is a collection of fewer than $\mathfrak{c}$ $\uparrow$-dense, $\uparrow$-open subsets of P, then there is a $\mathcal{D}$-generic subset of P.

MA′: If $(P, \leq)$ is an $\uparrow$-ccc poset and $\mathcal{D}$ is a collection of fewer than $\mathfrak{c}$ $\uparrow$-dense subsets of P, then there is a $\mathcal{D}$-generic subset of P.

MA″: If $(P, \leq)$ is an $\uparrow$-ccc poset and $\mathcal{D}$ is a collection of fewer than $\mathfrak{c}$ $\uparrow$-dense subsets of P, then there is a $\uparrow$-directed subset of P.

Proposition 10. *MA is equivalent to MA′.*

Proof. Certainly, MA′ implies MA, since the hypothesis of MA′ is weaker than that of MA but the conclusions are identical.

It remains to show MA implies MA′. To that end, let $\mathcal{D}$ be as in the hypothesis of MA′. For each $D \in \mathcal{D}$, let

$$D^\uparrow = \{p \in P \mid d \leq p \text{ for some } d \in D\}$$

and let

$$\mathcal{D}^{\uparrow} = \{D^{\uparrow} \mid D \in \mathcal{D}\}.$$

Each $D^{\uparrow}$ is $\uparrow$-open, so $\mathcal{D}^{\uparrow}$ is a collection of fewer than $\mathfrak{c}$ $\uparrow$-dense, $\uparrow$-open subsets of P. By MA, there is a $\mathcal{D}^{\uparrow}$-generic subset G of P.

The proof is finished by showing G is also $\mathcal{D}$-generic. Toward that goal, choose any $D \in \mathcal{D}$. Since G is $\mathcal{D}^{\uparrow}$-generic, there exists an element $p \in D^{\uparrow} \cap G$. By the definition of $D^{\uparrow}$, there exists an element $d \in D$ such that $d \leq p$. Since G is $\downarrow$-open, it follows that $d \in G$, and so $D \cap G \neq \emptyset$. As D was chosen arbitrarily from $\mathcal{D}$, it follows that G is $\mathcal{D}$-generic, as desired. □

Proposition 11. *MA is equivalent to MA″.*

Proof. Certainly, MA implies MA″, since the conclusion of MA″ is weaker than that of MA but the hypothesis are identical.

It remains to show MA″ implies MA. To that end, let $\mathcal{D}$ be as in the hypothesis of MA. By MA″, there is an $\uparrow$-directed subset U of P. Define

$$U^{\downarrow} = \{p \in P \mid p \leq u \text{ for some } u \in U\}.$$

This definition guarantees $U^{\downarrow}$ is $\downarrow$-open. All that remains is to check that it is $\uparrow$-directed. To that end, let $p_1, p_2 \in U^{\downarrow}$. By the definition of $U^{\downarrow}$, there are elements $u_1, u_2 \in U$ such that $p_1 \leq u_1$ and $p_2 \leq u_2$. Thus, $p_1 \vee p_2 \leq u_1 \vee u_2$. Now, $u_1 \vee u_2 \in U (\subseteq U^{\downarrow})$ since U is $\uparrow$-directed. Since $U^{\downarrow}$ is $\downarrow$-open, $p_1 \vee p_2 \in U^{\downarrow}$. Therefore, $U^{\downarrow}$ is $\uparrow$-directed, as desired. □