Math 731 Homework

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June 14, 2012

1 Problem 15B1

Definition 1.1. *A space X is said to be* completely normal *if every subspace of X is normal.*

Proposition 1.2. *X is completely normal if and only if, whenever A and B are subsets of X with $A \cap \mathrm{Cl}_X(B) = \mathrm{Cl}_X(A) \cap B = \emptyset$, then there are disjoint open sets $U \supset A$ and $V \supset B$.*

Proof. ($\Rightarrow$) Let A and B be given with $A \cap \mathrm{Cl}_X(B) = \mathrm{Cl}_X(A) \cap B = \emptyset$. Denote by Y the subspace $X - (\mathrm{Cl}_X(A) \cap \mathrm{Cl}_X(B))$. Now, both A and B are contained in Y. To see this, observe that

$$\begin{aligned} A \cap (\mathrm{Cl}_X(A) \cap \mathrm{Cl}_X(B)) &= (A \cap \mathrm{Cl}_X(B)) \cap \mathrm{Cl}_X(A) \\ &= \emptyset \cap \mathrm{Cl}_X(A) \\ &= \emptyset. \end{aligned}$$

Similarly,

$$\begin{aligned} B \cap (\mathrm{Cl}_X(A) \cap \mathrm{Cl}_X(B)) &= (\mathrm{Cl}_X(A) \cap B) \cap \mathrm{Cl}_X(B) \\ &= \emptyset \cap \mathrm{Cl}_X(B) \\ &= \emptyset. \end{aligned}$$

Next, we claim that $\mathrm{Cl}_Y(A)$ and $\mathrm{Cl}_Y(B)$ are disjoint, closed sets in Y. For disjointness, obvsere that

$$\begin{aligned} \mathrm{Cl}_Y(A) \cap \mathrm{Cl}_Y(B) \cap Y &\subset \mathrm{Cl}_X(A) \cap \mathrm{Cl}_X(B) \cap Y \\ &= \emptyset. \end{aligned}$$

For closedness, note that the closure of a set in a topological space is always closed in that space.

By the complete normality of X, we have that Y is normal. Hence, there exist sets U and V that are open in Y (and so open in X) with $U \supset \mathrm{Cl}_Y(A)$ and $V \supset \mathrm{Cl}_Y(B)$. As A and B are both contained in Y, we further have that $\mathrm{Cl}_Y(A) \supset A$ and $\mathrm{Cl}_Y(B) \supset B$. Therefore, $U \supset A$ and $V \supset B$, as desired.

($\Leftarrow$) Let Y be a subspace of X and let A and B be disjoint closed subsets of Y. Observe that

$$\begin{aligned} A \cap \mathrm{Cl}_X(B) &= A \cap B \\ &= \emptyset, \end{aligned}$$

and similarly

$$\begin{aligned} \mathrm{Cl}_X(A) \cap B &= A \cap B \\ &= \emptyset. \end{aligned}$$

By hypothesis, there are disjoint open subsets U and V of X with $U \supset A$ and $V \supset B$. It follows that the sets $U \cap Y$ and $V \cap Y$ are disjoint open subsets of Y with $(U \cap Y) \supset A$ and $(V \cap Y) \supset B$. Hence, Y is normal, and so X is completely normal. □

2 Problem 15B3

Lemma 2.1. *If M is metrizable and $N \subset M$, then the subspace N is metrizable with the topology generated by the restriction of any metric which generates the topology on M.*

Proof. Let τ be the topology on M generated by a metric ρ. Let σ be the relative topology on N and let ρ_N be the restriction of ρ to N. We show that σ is generated by ρ_N.

Let $O \in \sigma$. It must be that $O = N \cap G$ for some $G \in \tau$. Since M is generated by ρ, we know that $G = \bigcup_{x \in G} B_\rho(x, \epsilon_x)$, where $\epsilon_x > 0$ may depend on x. Now,

$$\begin{aligned}
O &= N \cap G \\
&= N \cap \bigcup_{x \in G} B_\rho(x, \epsilon_x) \\
&= \bigcup_{x \in G} N \cap B_\rho(x, \epsilon_x) \\
&= \bigcup_{x \in N \cap G} B_{\rho_N}(x, \epsilon_x).
\end{aligned}$$

Hence, O is the union of open balls with respect to the metric ρ_N. Therefore, σ is generated by the ρ_N, as desired. □

Proposition 2.2. *Every metric space is completely normal.*

Proof. By the lemma, any subspace of a metric space is itself a metric space. As every metric space is T_4 (hence, normal), it follows that every subspace of a metric space is normal. That is, every metric space is completely normal. □

3 Problem 16B2

Proposition 3.1. *Any base for the open sets in a second countable space has a countable subfamily that is a base.*

Proof. Let X be a second countable space, $\mathcal{B} = \{B_\alpha \mid \alpha \in \Gamma\}$ be any base for X, and $\mathcal{C} = \{C_i \mid i \in \mathbb{N}\}$ be the countable base guaranteed by the second countability of X. Our aim is to show that, for each $i \in \mathbb{N}$, C_i can be represented as the union of countably-many members of $\mathcal{B}$ (call this countable subcollection $\mathcal{B}_{C_i}$). Since the union of countably-many sets each having countably-many members is again countable, the set $\{A \mid A \in \mathcal{B}_{C_i} \text{ for some } i\}$ will be a countable subfamily of $\mathcal{B}$ that is a base for X.

To finish the proof, let $C_k \in \mathcal{C}$. Since $\mathcal{B}$ is a base for X, we can write $C_k = \bigcup_{i \in I} B_i$ for some subset I of the indexing set Γ. Now, for each $x \in C_k$, pick a set $B_{i_x} \in \mathcal{B}$ with $x \in B_{i_x}$ and $i_x \in I$. Since $\mathcal{C}$ is also a base for X, we can find some C_{i_x} with $x \in C_{i_x} \subset B_{i_x}$. It follows that $\{B_{i_x} \mid x \in C_k\}$ is a countable set (we chose one element of $\mathcal{B}$ for each element of the countable set $\mathcal{C}$) whose union is C_i (by construction, every element of C_i is present in the union and every B_{i_x} is contained in C_i). □

4 Problem 16B3

Proposition 4.1. *Any increasing chain of real numbers that is well ordered by the usual order must be countable.*

Proof. Let A be a set of real numbers that is well-ordered by the usual order. We claim that, for each $a \in A$, there is $n_a \in \mathbb{N}$ such that $\left(a, a + \frac{1}{n_a}\right] \cap A = \emptyset$. Observe that the truth of this claim implies the countability of A, as each interval will contain a distinct rational number. Suppose now, for the purpose of contradiction, that the claim does not hold. That is, there exists some $b \in A$ such that, for all $n \in \mathbb{N}$, $\left(b, b + \frac{1}{n}\right] \cap A \neq \emptyset$. It follows that the set $A \setminus \{c \in A \mid c \leq b\}$ has no least element, which is a contradiction with the fact that A is well-ordered, thus completing the proof. □

5 The One-Point Compactification: Construction

The procedure used to to obtain the one-point compactification X^* of a locally compact, non-compact Hausdorff space X can be applied to any space Y. That is, $Y^* = Y \cup \{p\}$ with neighborhoods $y \in Y$ unchanged in Y^* while neighborhoods of p have the form $\{p\} \cup (Y - L)$, where L is a subset of Y with compact closure. Y^* is called the *Alexandroff extension* of Y.

6 Problem 19A1

Proposition 6.1. *The assignment of neighborhoods in Y^* described above is valid.*

Proof. Recall that one can define a topology on a set Y by specifying, for each $x \in Y$, a set $\mathcal{U}_x$ (called the *neighborhood system at x*) satisfying

1. If $U \in \mathcal{U}_x$, then $x \in U$.
2. If $U, V \in \mathcal{U}_x$, then $U \cap V \in \mathcal{U}_x$.
3. If $U \in \mathcal{U}_x$, then there is a set $V \in \mathcal{U}_x$ such that $U \in \mathcal{U}_y$ for each $y \in V$.
4. If $U \in \mathcal{U}_x$ and $U \subset V$, then $V \in \mathcal{U}_x$.

Using this approach, we call a set open whenever it contains a neighborhood of each of its points.

Observe that, since Y is a topological space, there already exists a neighborhood system satisfying these requirements for each $x \in Y$. Let $\mathcal{U}_p$ contain all sets of the form $\{p\} \cup (Y - L)$, where L is a subset of Y with compact closure. We proceed by verifying that $\mathcal{U}_p$ satisfies the above requirements.

Claim 6.2. *If $U \in \mathcal{U}_p$, then $p \in U$.*

Proof. By definition, $U = \{p\} \cup (Y - L)$ (some L). Hence, $p \in U$. □

Claim 6.3. *If $U, V \in \mathcal{U}_p$, then $U \cap V \in \mathcal{U}_p$.*

Proof. Let $U = \{p\} \cup (Y - L)$ and $V = \{p\} \cup (Y - K)$, where L and K are subsets of Y with compact closure. It follows that

$$\begin{aligned} U \cap V &= [\{p\} \cup (Y - L)] \cap [\{p\} \cup (Y - K)] \\ &= \{p\} \cup (Y - (L \cup K)). \end{aligned}$$

Now, $\mathrm{Cl}_Y(L \cup K) = \mathrm{Cl}_Y(L) \cup \mathrm{Cl}_Y(K)$, each of which is compact by assumption. As the union of two compact spaces is again compact, we conclude that $L \cup K$ is indeed a subset of Y with compact closure. Hence, $U \cap V \in \mathcal{U}_p$. □

Claim 6.4. *If $U \in \mathcal{U}_p$, then there is a set $V \in \mathcal{U}_p$ such that $U \in \mathcal{U}_y$ for each $y \in V$.*

Proof. Let $U \in \mathcal{U}_p$ and choose any open $V \in \mathcal{U}_p$ with $V \subset U$. Since V is open, there exists, for all $y \in V$, an open subset G of V with $y \in G$ and $p \notin G$. Hence, $G \in \mathcal{U}_y$. Since $y \in Y$, $\mathcal{U}_y$ satisfies the property that any superset of G is also contained in $\mathcal{U}_y$. In particular, $U \in \mathcal{U}_y$, as desired. □

Claim 6.5. *If $U \in \mathcal{U}_p$ and $U \subset V$, then $V \in \mathcal{U}_p$.*

Proof. Let $U = \{p\} \cup (Y - L)$ for some subset L of Y with compact closure. It follows that $V = \{p\} \cup (Y - K)$ with $K \subset L$, which in turn gives that $\mathrm{Cl}_Y(K) \subset \mathrm{Cl}_Y(L)$. As a closed subset of compact space is compact, we have that K has compact closure. Hence, $V \in \mathcal{U}_p$. □

Therefore, the neighborhood systems as defined indeed give a topology on Y^*. We conclude by showing that its relative topology on Y is the original topology. To that end, let $\sigma = \{G \mid G \text{ open in } Y\}$ and $\tau = \{G \cap Y \mid G \text{ open in } Y^*\}$. Evidently, $\sigma \subset \tau$. We also have that, for any open G in Y^*,

$$\begin{aligned} G \cap Y &= [\{p\} \cup (Y - L)] \cap Y \qquad \text{(some } L \text{ with compact closure in } Y\text{)} \\ &= Y - L. \end{aligned}$$

(Do we here require the stronger notion that L be compact rather than merely having compact closure? For example, $(0,1)$ has compact closure in $\mathbb{R}$, yet $\mathbb{R} - (0,1)$ is not open.)

Pending the resolution of the above parenthetical remark, we will have shown that $\sigma = \tau$, and so the relative topology on Y is precisely the original topology. □

7 Problem 19A4

Proposition 7.1. *Y^* is Hausdorff if and only if Y is locally compact and Hausdorff.*

Proof. ($\Rightarrow$) Let $x, y \in Y \subset Y^*$. Since Y^* is Hausdorff, we can find disjoint open neighborhoods U and V in Y^* with $x \in U$ and $y \in V$. If neither U nor V contains p, then U and V suffice to show that Y is Hausdorff. Otherwise, let it be that $U = \{p\} \cup (Y - L)$ for some L with compact closure in Y (it cannot be that both U and V are of this form, as this would violate disjointness). Define the set $U' = Y - Cl_Y(L)$. We have that $x \in U'$, $y \in V$ with U' and V disjoint and open.

To see that Y is locally compact, let $x \in Y$ be given. Since Y^* is Hausdorff, there is an open set U of Y^* such that $x \notin U$. Now, $U = \{p\} \cup (Y - L)$ for some subset L of Y with compact closure. Hence, $x \in L \subset \mathrm{Cl}_Y(L)$, which is compact. Since Y is Hausdorff and every point of Y has a compact neighborhood, it follows that Y is locally compact.

($\Leftarrow$) Let $x, y \in Y^*$. If neither x nor y is p, then the fact that Y is Hausdorff implies that x and y can be put into disjoint open neighborhoods. Suppose now that $y = p$. Choose, by the local compactness of Y, some compact neighborhood $V \subset Y$ of x. It follows that $\mathrm{Int}_Y(V)$ is an open set containing x. We also have that $\mathrm{Int}_Y(Y^* - V) = \mathrm{Int}_Y(\{p\} \cup (Y - V))$, which is an open neighborhood (as it is a member of the base) containing p. Thus, for any $x, y \in Y^*$, we have produced two disjoint open sets containing them. Hence, Y^* is Hausdorff. $\square$

8 Problem 16D1

Proposition 8.1. *The Sorgenfrey line* $\boldsymbol{E}$ *is Lindelöf.*

Proof. Let $\mathcal{C}$ be a basic open (in $\mathbf{E}$) cover of $\mathbb{R}$. That is, $\mathcal{C} = \{[x_\alpha, y_\alpha) \mid x_\alpha, y_\alpha \in \mathbb{R}, \alpha \in \Gamma\}$. Let $\mathcal{C}' = \{(x_\alpha, y_\alpha) \mid [x_\alpha, y_\alpha) \in \mathcal{C}\}$. Finally, let $A = \bigcup C'$. Evidently, $\mathcal{C}'$ covers A. Now, observe that $\mathcal{C}'$ contains open sets from the usual topology on $\mathbb{R}$. Since $\mathbb{R}$ with the usual topology is second-countable, it is Lindelöf. Hence, there is a countable subcollection of $\mathcal{C}'$ (and so of $\mathcal{C}$) covering A.

Next, we claim that A misses only countably-many points of $\mathbb{R}$. To see this, let x and y be distinct real numbers that are not elements of A. Without loss of generality, let $x < y$. Since $\mathcal{C}$ covers $\mathbb{R}$, it follows that x and y are left endponts of distinct intervals of $\mathcal{C}$. Denote these intervals by $[x, x')$ and $[y, y')$, respectively. Evidently, $[x, x')$ and $[y, y')$ are disjoint (if not, then $y \in (x, x')$, and so $y \in A$). Hence, we can identify with each of x and y a distinct rational number belonging to (x, x') and (y, y'), respectively. Thus, A misses only countably-many elements of $\mathbb{R}$, and so there is a countable subcollection of $\mathcal{C}$ covering A.

We have shown that $\mathcal{C}$ admits a countable subcover for both A and $\mathbb{R} \setminus A$. Taking these subcovers together yields the desired subcover of $\mathbb{R}$. $\square$

Corollary 8.2. *The Sorgenfrey line is a* T_4*-space.*

Proof. Since the Soregenfrey line is regular and Lindelöf, it is normal. Furthermore, it is T_1. Taken together, we have that the Sorgenfrey line is T_4. $\square$

9 Problem 16D2

Proposition 9.1. *Every uncountable subset of a Lindelöf space contains an accumulation point.*

Proof. We establish the contrapositive. To that end, let A be a subset of a Lindelöf space X such that A contains no accumulation points. It follows that, for each $x \in X$, there is an open neighborhood U_x of x with $|U_x \cap A|$

finite. Let $\mathcal{C}$ denote the collection $\{U_x \mid x \in X\}$. Since $\mathcal{C}$ is an open cover of the Lindelöf space X, $\mathcal{C}$ admits a countable subcover $\mathcal{C}'$ of X (and so of A). Since $|A \cap U|$ is finite for all U belonging to the countable collection $\mathcal{C}'$, it follows that A is countable, thus establishing the contrapositive. □

10 Extra Problem

Proposition 10.1. *A nonempty collection of subsets of X is a subbase for some filter on X if and only if it has the finite intersection property.*

Proof. ($\Rightarrow$) We establish the contrapositive. That is, suppose $\mathcal{A}$ is a nonempty collection of subsets of X that fails to satisfy the finite intersection property. It follows that there are sets $A_1, \ldots, A_k \in \mathcal{A}$ such that $\bigcap_{i=1}^k A_i = \emptyset$. Consider now the collection $\mathcal{C} = \{\bigcap_{i=1}^n A_i \mid A_i \in \mathcal{A} \text{ for all } i\}$. Evidently, $\emptyset \in \mathcal{C}$, and so it cannot be a base for any filter on X (as no filter contains the empty set). Hence, $\mathcal{A}$ is not a subbase for any filter on X, thus establishing the contrapositive.

($\Leftarrow$) Let $\mathcal{S}$ be a nonempty collection of subsets of X having the finite intersection property. Define $\mathcal{B}$ to be the collection

$$\{F \subset X \mid \bigcap \mathcal{S}' \subset F \text{ for some finite subcollection } \mathcal{S}' \text{ of } \mathcal{S}\}.$$

Observe first that $\mathcal{B}$ is a nonempty collection of nonempty subsets of X, as $\mathcal{S}$ satisfies the finite intersection property. We further claim that $\mathcal{B}$ is in fact a base for a filter on X, and so $\mathcal{S}$ will be a subbase for this filter. To that end, let $B_1, B_2 \in \mathcal{B}$. By definition of $\mathcal{B}$,

$$\begin{aligned} B_1 \cap B_2 &= \left(\bigcap \mathcal{S}_1\right) \cap \left(\bigcap \mathcal{S}_2\right) \qquad \text{(some } \mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{S}\text{)} \\ &= \bigcap (\mathcal{S}_1 \cup \mathcal{S}_2). \end{aligned}$$

Evidently, $\mathcal{S}_1 \cup \mathcal{S}_2$ is a finite subcollection of $\mathcal{S}$, as both $\mathcal{S}_1$ and $\mathcal{S}_2$ are. Hence, $B_1 \cap B_2 \in \mathcal{B}$, and so $\mathcal{B}$ is a base for the filter $\{F \subset X \mid B \subset F \text{ for some } B \in \mathcal{B}\}$. □

11 Problem 12B1

Proposition 11.1. *The intersection of any number of filters on X is a filter on X.*

Proof. Let $\mathcal{F}$ be the intersection of filters $\mathcal{F}_\alpha$ for $\alpha \in \Gamma$. We claim that $\mathcal{F}$ is itself a filter.

First, observe that $\emptyset \notin \mathcal{F}$ and $X \in \mathcal{F}$, since this is the case for each of the $\mathcal{F}_\alpha$.

Next, let F_1 and F_2 be elements of $\mathcal{F}$. It follows that F_1 and F_2 belong to $\mathcal{F}_\alpha$ for all $\alpha \in \Gamma$, and so $F_1 \cap F_2 \in \mathcal{F}_\alpha$ for all $\alpha \in \Gamma$. Hence, $F_1 \cap F_2 \in \mathcal{F}$.

Finally, let $F \in \mathcal{F}$ and let G be any set containing F. Since $F \in \mathcal{F}_\alpha$ for all $\alpha \in \Gamma$, it follows that $G \in \mathcal{F}_\alpha$ for all $\alpha \in \Gamma$. Hence, $G \in \mathcal{F}$.

Taken together, the above three observations verify that $\mathcal{F}$ is indeed a filter, as desired. □

12 Problem 12B3

Proposition 12.1. *Every filter is the intersection of the ultrafilters containing it.*

Proof. Let $\mathcal{F}$ be a filter and let $\mathcal{U} = \bigcap_{\alpha \in \Gamma} \mathcal{U}_\alpha$, where the $\mathcal{U}_\alpha$ constitute all ultrafilters containing $\mathcal{F}$. Certainly, $\mathcal{F} \subset \mathcal{U}$. Suppose, for the purpose of contradiction, that this inclusion is strict. It follows that there is some set A with $A \notin \mathcal{F}$ and $A \in \mathcal{U}$. Consider the filter $\mathcal{G}$ generated by $\mathcal{F} \cup \{A^c\}$ (this is indeed a filter, since $A \notin \mathcal{F}$ and so $\emptyset \notin \mathcal{G}$). We have that $\mathcal{F} \subset \mathcal{G} \subset \mathcal{U}_\beta$, for some $\beta \in \Gamma$. Now, since $A \in \mathcal{U}$, $A \in \mathcal{U}_\beta$, but also $A^c \in \mathcal{U}_\beta$, which is a contradiction. Therefore, it must be that $\mathcal{F} = \mathcal{U}$. □

13 Problem 16C1

Definition 13.1. *Let $\aleph$ be any cardinal number. A space X is said to have <u>caliber</u> $\aleph$ if, whenever $\mathcal{U}$ is a family of open subsets of X with $|\mathcal{U}| = \aleph$, then there exists a subfamily $\mathcal{V}$ of $\mathcal{U}$ with $|\mathcal{V}| = \aleph$ and $\bigcap \mathcal{V} \neq \emptyset$.*

Proposition 13.2. *Every separable space has caliber $\aleph_1$.*

Proof. Let X be a separable space having countable dense subset D. Let $\mathcal{U}$ be a collection of open subsets of X with $|\mathcal{U}| = \aleph_1$. As D is dense in X, each open set of $\mathcal{U}$ contains a point of D. It follows that there exists $x \in D$ belonging to uncountably-many open sets of $\mathcal{U}$. (Were this not the case, each of the elements of the countable set D appears in only countably-many open sets of $\mathcal{U}$, which would imply that $\mathcal{U}$ is countable.) Let $\mathcal{V} = \{U \in \mathcal{U} \mid x \in U\}$. Evidently, $|\mathcal{V}| = \aleph_1$ and $x \in \bigcap \mathcal{V}$. Therefore, X has caliber $\aleph_1$. □

14 Problem 16C3

Definition 14.1. *We say X satisfies the* <u>*countable chain condition*</u> *provided every family of disjoint open subsets of X is countable.*

Proposition 14.2. *If X has caliber $\aleph_1$, then X satisfies the countable chain condition.*

Proof. We establish the contrapositive. To that end, suppose that X does not satisfy the countable chain condition. It follows that there is an uncountable family $\mathcal{U}$ of disjoint open sets of X. By the Axiom of Choice, we may assume that $|\mathcal{U}| = \aleph_1$ (if $|\mathcal{U}| > \aleph_1$, we use the Axiom of Choice to reduce it to a collection of size $\aleph_1$). Now, since $\mathcal{U}$ is a collection of disjoint sets, it follows that $\bigcap \mathcal{U} = \emptyset$, and so X does not have caliber $\aleph_1$. □

15 Problem 16D3

Proposition 15.1. *A regular space is Lindelöf if and only if each open cover has a countable subcollection whose closures cover (i.e. has a* <u>*countable dense subsystem*</u>*).*

Proof. ($\Rightarrow$) Let X be a Lindelöf space. Any open cover of X admits a countable subcover $\mathcal{U}$. Evidently, $U \subset \overline{U}$ for all $U \in \mathcal{U}$, and so the set $\{\overline{U} \mid U \in \mathcal{U}\}$ is also a countable cover of X.

($\Leftarrow$) Let X be a regular space and let $\mathcal{U}$ be an open cover of X. For each $x \in X$, let U_x be an open set in $\mathcal{U}$ containing x. Since X is regular, there is an open set V_x containing x with $\overline{V_x} \subset U_x$. Let $\mathcal{V}$ be the collection of sets V_x for all $x \in X$. Evidently, $\mathcal{V}$ is also a cover of X. By hypothesis, $\mathcal{V}$ admits a countable subcollection $\mathcal{V}' = \{V_i \mid i \in \mathbb{N}\}$ such that

$$\bigcup_{i \in \mathbb{N}} \overline{V_i} = X.$$

By construction, each $\overline{V_i}$ is contained in some open set (call it U_i) belonging to $\mathcal{U}$. It follows that the collection $\{U_i \mid i \in \mathbb{N}\}$ is a countable subcollection of $\mathcal{U}$ covering X. Therefore, X is Lindelöf. □

16 Problem 17C1

Definition 16.1. *A compact space X is maximal compact provided every strictly finer topology on X is noncompact.*

Proposition 16.2. *A compact space X is maximal compact if and only if every compact subset is closed.*

Proof. ($\Rightarrow$) We establish the contrapositive. To that end, let (X, τ) be a compact space having a compact subset K that is not closed. Furthermore, let $\mathcal{B}$ be a base for the open sets of X.

Now, since K is not closed, $X - K$ is not open. Let τ' be the topology generated by $\tau \cup \{X - K\}$. Evidently, τ' is a strictly finer topology than τ.

We claim next that (X, τ') is compact. To that end, let $\mathcal{C}$ be an open cover (using sets from τ') of X. If $X - K \notin \mathcal{C}$, then only open sets of τ appear in the cover. Since X is compact under τ, we can find a finite subcover of X. Suppose then that $X - K \in \mathcal{C}$. Since K is compact under τ and K is disjoint from $X - K$, we can find a finite subcover of K using elements of $\mathcal{C} - \{X - K\}$. Taking this cover of K together with $X - K$ gives the desired cover of X.

In either case, we conclude that (X, τ) is not maximal, thus establishing the contrapositive.

($\Leftarrow$) We establish the contrapositive. To that end, suppose that (X, τ) is not maximal compact. Let τ' denote a strictly finer compact topology than τ. Since τ' is strictly finer than τ, we can find a closed set F belonging to τ' but not belonging to τ. Since F is a closed (under τ') subset of a compact space, F is compact under τ'. Hence, every open cover of F by open sets of τ' admits a finite subcover. In particular, any open cover of F using only open sets of τ admits a finite subcover, and so F is compact under τ. Hence, under τ, F is a compact subset of X that is not closed, thus establishing the contrapositive. $\square$

17 Problem 17F4

Proposition 17.1. *Let $X_1, X_2, \ldots$ all be first countable. The product $\prod X_n$ is countably compact if and only if each X_n is countably compact.*

Proof. ($\Rightarrow$) Let $\prod X_n$ be countably compact. That is, every countable open cover of $\prod X_n$ admits a finite subcover. For a fixed k, let $\mathcal{C}$ be a countable open cover of X_k. For each open $C \in \mathcal{C}$, let $U_C = \prod U_i$. where $U_k = C$ and $U_i = X_i$ for $i \neq k$. Denote by $\mathcal{D}$ the collection $\{U_C \mid C \in \mathcal{C}\}$. Evidently, $\mathcal{D}$ is a countable open cover of $\prod X_n$, which admits a finite subcover $\mathcal{D}'$ by the countable compactness of $\prod X_n$. It follows that the collection of all U_k from the open sets belonging to $\mathcal{D}'$ (that is, the collection containing the factor corresponding to X_k from each open set in $\mathcal{D}'$) is a finite subcollection of $\mathcal{C}$ covering X_k, and so X_k is countably compact. As k was arbitrary, it follows that each of the product spaces is countably compact.

($\Leftarrow$) As every countably compact space is sequentially compact, it suffices to show that $\prod X_n$ is sequentially compact. To that end, let $\langle x_n \rangle$ be a sequence in the product space. For each i, $\langle x_n(i) \rangle$ is a sequence in X_i. Since X_i is first countable and countably compact, it is sequentially compact. Hence, $\langle x_n(i) \rangle$ has a convergent subsequence $\langle x_{n_k}(i) \rangle$. (This is not itself enough to conclude that the sequence in the product has a convergent subsequence, as, in general, the product of sequentially compact spaces need not be sequentially compact. Hence, I have to make more explicit use of first countability, but I do not see how to do this.) □

18 Problem 17F5

Proposition 18.1. *Continuous images of countably compact spaces are countably compact.*

Proof. Let f be a continuous function from the countably compact space X onto Y and let $\mathcal{C}$ be a countable open cover of Y. By the continuity of f, the set $\{f^{-1}(C) \mid C \in \mathcal{C}\}$ is an open cover of X. Since X is countably compact, $\{f^{-1}(C) \mid C \in \mathcal{C}\}$ admits a finite subcover $\{f^{-1}(C_1), \ldots, f^{-1}(C_n) \mid C_i \in \mathcal{C}\}$ of X. Since f is onto, it follows that the C_i cover Y. Hence, there is a finite subcover of the countable cover $\mathcal{C}$, and therefore Y is countably compact. □

Proposition 18.2. *Closed subspaces of countably compact spaces are countably compact.*

Proof. Let X be countably compact, F a closed subspace of X, and $\mathcal{C}$ a countable open cover of F. For each $C \in \mathcal{C}$, there is an open set $U_C \in X$ such that $U_C \cap F = C$. Now, since F is closed, $X - Y$ is open in X. Hence, $\{X - F\} \cup \{U_C \mid C \in \mathcal{C}\}$ is an open cover of X, which by the countable compactness of X admits a finite subcover $\{X - F\} \cup \{U_1, \ldots, U_n \mid U_i \in X\}$.

It follows that $\{U_1 \cap F, \ldots, U_n \cap F\}$ is a finite subcollection of $\mathcal{C}$ covering F. ∎

19 Problem 22F1

Lemma 19.1. *Let x and y be real numbers both strictly greater than -1. If $x \leq y$, then $\frac{x}{1+x} \leq \frac{y}{1+y}$.*

Proof. We have that

$$\begin{aligned} x &\leq y \\ x + xy &\leq y + xy \\ x(1+y) &\leq y(1+x) \\ \frac{x}{1+x} &\leq \frac{y}{1+y} && \text{(since } x, y \text{ strictly greater than } -1\text{).} \end{aligned}$$

∎

Remark 19.2. *One can think of the previous lemma as follows: Suppose we x mL of salt and y mL of salt and we add them each to separate 1 mL containers of water. The ratio $\frac{x}{1+x}$ measures the concentration of salt in the first container and $\frac{y}{1+y}$ measures the concentration of salt in the second container. Since x is a smaller amount of salt than y, then surely the first container is less concentrated than the second. That is, $\frac{x}{1+x} \leq \frac{y}{1+y}$.*

Proposition 19.3. *If ρ is a metric on X, then both*

$$\rho_1(x,y) = \min\{1, \rho(x,y)\}$$

and

$$\rho_2(x,y) = \frac{\rho(x,y)}{1+\rho(x,y)}$$

are metrics equivalent to ρ on X.

Proof. We first establish that ρ_1 and ρ_2 are indeed metrics on X. In what follows, let x, y, z be arbitrary points in X.

For the positive definiteness of ρ_1, observe that

$$\begin{aligned} \rho_1(x,y) &= \min\{1, \rho(x,y)\} \\ &\geq 0 && \text{(since } \rho \text{ is positive definite),} \end{aligned}$$

with equality if and only if $\rho(x,y) = 0$, which occurs if and only if $x = y$ (since ρ is positive definite).

For the symmetry of ρ_1, observe that

$$\begin{aligned}\rho_1(x,y) &= \min\{1, \rho(x,y)\} \\ &= \min\{1, \rho(y,x)\} && \text{(since } \rho \text{ is symmetric)} \\ &= \rho_1(y,x).\end{aligned}$$

For the subadditivity of ρ_1, observe that

$$\begin{aligned}\rho_1(x,y) &= \min\{1, \rho(x,y)\} \\ &\leq \min\{1, \rho(x,z) + \rho(z,y)\} && \text{(since } \rho \text{ is subadditive)} \\ &\leq \min\{1, \rho(x,z)\} + \min\{1, \rho(z,y)\} \\ &= \rho_1(x,z) + \rho(z,y).\end{aligned}$$

Therefore, ρ_1 is a metric on X.

For the positive definiteness of ρ_2, observe that

$$\begin{aligned}\rho_2(x,y) &= \frac{\rho(x,y)}{1 + \rho(x,y)} \\ &\geq 0 && \text{(since } \rho \text{ is positive definite),}\end{aligned}$$

with equality if and only if $\rho(x,y) = 0$, which occurs if and only if $x = y$ (since ρ is positive definite).

For the symmetry of ρ_2, observe that

$$\begin{aligned}\rho_2(x,y) &= \frac{\rho(x,y)}{1 + \rho(x,y)} \\ &= \frac{\rho(y,x)}{1 + \rho(y,x)} && \text{(since } \rho \text{ is symmetric)} \\ &= \rho_2(y,x).\end{aligned}$$

For the subadditivity of ρ_2, observe that

$$\begin{aligned}\rho(x,y) &\leq \rho(x,z) + \rho(z,y) && \text{(since } \rho \text{ is subadditive)} \\ \frac{\rho(x,y)}{1 + \rho(x,y)} &\leq \frac{\rho(x,z) + \rho(z,y)}{1 + \rho(x,z) + \rho(z,y)} && \text{(by lemma).}\end{aligned}$$

The lefthand side of the inequality is precisely $\rho_2(x, y)$, and so we have that

$$\begin{aligned}
\rho_2(x,y) &\leq \frac{\rho(x,z)+\rho(z,y)}{1+\rho(x,z)+\rho(z,y)} \\
&= \frac{\rho(x,z)}{1+\rho(x,z)+\rho(z,y)} + \frac{\rho(z,y)}{1+\rho(x,z)+\rho(z,y)} \\
&\leq \frac{\rho(x,z)}{1+\rho(x,z)} + \frac{\rho(z,y)}{1+\rho(z,y)} \qquad \text{(since } \rho \text{ is positive definite)} \\
&= \rho_2(x,z)+\rho_2(z,y).
\end{aligned}$$

Therefore, ρ_2 is a metric on X.

We show next that ρ and ρ_1 are equivalent by showing (X, ρ) and (X, ρ_1) have the same open sets. To that end, consider the basic open set $B_\rho(x, \epsilon)$ of (X, ρ). We have that

$$B_\rho(x,\epsilon) = \bigcup_{y \in B_\rho(x,\epsilon)} B_\rho(y, \epsilon_y),$$

where each ϵ_y depends on y. By definition of ρ_1, we have, for each y,

$$B_{\rho_1}(y, \epsilon_y) \subset B_\rho(y, \epsilon_y).$$

Taken together, we see that

$$B_\rho(x,\epsilon) = \bigcup_{y \in B_\rho(x,\epsilon)} B_{\rho_1}(y, \epsilon_y),$$

and so $(X, \rho) \subset (X, \rho_1)$.

Next, consider the basic open set $B_{\rho_1}(x, \epsilon)$ of (X, ρ_1). As before,

$$B_{\rho_1}(x,\epsilon) = \bigcup_{y \in B_{\rho_1}(x,\epsilon)} B_{\rho_1}(y, \epsilon_y)$$

where each ϵ_y depends on y. In particular, we may insist that $0 < \epsilon_y \leq 1$ for all y. With this restriction, we have, for each y,

$$B_{\rho_1}(y, \epsilon_y) = B_\rho(y, \epsilon_y).$$

Taken together, we see that

$$B_{\rho_1}(x,\epsilon) = \bigcup_{y \in B_{\rho_1}(x,\epsilon)} B_\rho(y, \epsilon_y),$$

and so $(X, \rho_1) \subset (X, \rho)$.

We show next that ρ and ρ_2 are equivalent by showing that (X, ρ) and (X, ρ_2) have the same open sets. That $(X, \rho) \subset (X, \rho_2)$ follows precisely as in the proof that $(X, \rho) \subset (X, \rho_1)$, as $B_{\rho_2}(x, \epsilon) \subset B_\rho(x, \epsilon)$ for all $x \in X$ and $\epsilon > 0$.

Next, consider the basic open set $B_{\rho_2}(x, \epsilon)$ of (X, ρ_2). As before,

$$B_{\rho_2}(x, \epsilon) = \bigcup_{y \in B_{\rho_2}(x,\epsilon)} B_{\rho_2}(y, \epsilon_y),$$

where each ϵ_y depends on y. Observe now that, for all y,

$$\begin{aligned} \rho_2(y, z) = \epsilon_y &\Rightarrow \frac{\rho(y, z)}{1 + \rho(y, z)} = \epsilon_y \\ &\Rightarrow \rho(y, z) = \epsilon_y(1 + \rho(y, z)) \\ &\Rightarrow \rho(y, z) = \frac{\epsilon_y}{1 - \epsilon_y}, \end{aligned}$$

where we insist that $0 < \epsilon_y < 1$ for all y. Taken together, we see that

$$B_{\rho_2}(x, \epsilon) = \bigcup_{y \in B_{\rho_2}(x,\epsilon)} B_\rho(y, \frac{\epsilon_y}{1 - \epsilon_y}),$$

and so $(X, \rho_2) \subset (X, \rho)$. □

20 Problem 22F2

Proposition 20.1. *Every metric generating the topology of a compact metrizable space is bounded.*

Proof. We proceed by establishing the contrapositive. To that end, let ρ be an unbounded metric generating a topology on a set X. Define, for all $n \in \mathbb{N}$, B_n to be the open set $B(x, n)$ for some fixed $x \in X$. The collection $\{B_n \mid n \in \mathbb{N}\}$ covers X but admits no finite subcover. To see this, let $\mathcal{C}$ be any finite subcollection of $\{B_n \mid n \in \mathbb{N}\}$. By construction, $\bigcup \mathcal{C} \subset B(x, M)$, where M denotes the maximum n such that $B_n \in \mathcal{C}$. Since ρ is unbounded, there is a point $y \in X$ with $\rho(x, y) > M$, and so $y \notin B(x, M)$. Hence, $\mathcal{C}$ does not cover X, and so X is not compact, thus establishing the contrapositive. □

21 Extra Problem 1

Lemma 21.1. *Every point of a closed ball in an ultrametric space is a center.*

Proof. Let (X, ρ) be an ultrametric space, and let $B(a, \epsilon)$ denote the closed ball around $a \in X$ of radius $\epsilon > 0$ (i.e. the set $\{y \in X \mid \rho(a, y) \leq \epsilon\}$). Let $b \in B(a, \epsilon)$ and let $x \in B(b, \epsilon)$. It follows that

$$\begin{aligned} \rho(a, x) &\leq \max\{\rho(a, b), \rho(b, x)\} \\ &\leq \epsilon, \end{aligned}$$

and so $x \in B(a, \epsilon)$. Hence, $B(b, \epsilon) \subset B(a, \epsilon)$, and a similar proof gives the reverse inclusion. Therefore, $B(a, \epsilon) = B(b, \epsilon)$, and so both a and b are centers. □

Proposition 21.2. *Every ultrametric space has a base of clopen sets.*

Proof. Let X be an ultrametric space and let $\mathcal{F}$ denote the collection of all closed balls in X. Evidently, $\mathcal{F}$ covers X. Observe next that, for any two balls of $\mathcal{F}$, either they are disjoint or one is contained in the other. To see this, let $F_1, F_2 \in \mathcal{F}$ with $x \in F_1 \cap F_2$ (such an x exists if F_1 and F_2 are not disjoint). Now, F_1 has the form $B(a, \epsilon_1)$ and F_2 has the form $B(b, \epsilon_2)$. Without loss of generality, let $\epsilon_1 \leq \epsilon_2$. By the lemma, $F_1 = B(a, \epsilon_1)$ and $F_2 = B(a, \epsilon_2)$, and so $F_1 \subset F_2$. From this fact, we conclude that $\mathcal{F}$ is a base for X (any F_1 and F_2 with nontrivial intersection has one containing the other).

Now, closed balls are certainly closed in this topology, as $\overline{F} = F$ for all $F \in \mathcal{F}$. To see that they are also open, observe that, by the lemma, F contains a basic neighborhood of each of its points (namely, F itself). Therefore, $\mathcal{F}$ is a base of clopen sets for X. □

22 Extra Problem 2

Proposition 22.1. *The metric given by*

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{2^n} & \text{otherwise, where } n = \max\{n \mid x(i) = y(i)\, \forall i \leq n\} \end{cases}$$

gives the product topology on $\mathbb{N}^{\mathbb{N}}$.

Proof. Let $\mathcal{B}$ denote the collection of all open balls of $\mathbb{N}^{\mathbb{N}}$. Evidently, $\mathcal{B}$ covers $\mathbb{N}^{\mathbb{N}}$. To see that it is, in fact, a base for $\mathbb{N}^{\mathbb{N}}$, let $B_1, B_2 \in \mathcal{B}$ and let $x \in B_1 \cap B_2$. Now, B_1 is of the form $B(a, \frac{1}{2^m})$ and B_2 is of the form $B(b, \frac{1}{2^n})$. Without loss of generality, let $m \leq n$. It follows that $x(i) = a(i) = b(i)$ for all $i \leq m$. Hence, $x \in B(x, \frac{1}{2^m}) \subset B_1 \cap B_2$, and so $\mathcal{B}$ is a base for $\mathbb{N}^{\mathbb{N}}$.

Now, the sets of $\mathcal{B}$ can be expressed as

$$B\left(x, \frac{1}{2^n}\right) = \prod_{i=1}^{n}\{x(i)\} \times \prod_{i=n+1}^{\infty} \mathbb{N}.$$

In other words, the basic open sets are products of open sets U_n of $\mathbb{N}$ where $U_n = \mathbb{N}$ for all but finitely-many n. Hence, $\mathcal{B}$ induces the product topology on $\mathbb{N}^{\mathbb{N}}$. □

23 Problem 17F1

Proposition 23.1. *A space is countably compact if and only if each sequence has a cluster point.*

Proof. ($\Rightarrow$) We proceed by establishing the contrapositive. To that end, suppose there is a sequence $\langle x_n \rangle$ having no cluster point in X. That is, for every $x \in X$, there exists a neighborhood (without loss of generality, an *open* neighborhood) N_x of x such that $N_x \setminus \{x\}$ contains no point of $\langle x_n \rangle$. Denote by N the open set $\bigcup\{N_x \mid x \text{ is not an element } \langle x_n \rangle\}$. By construction, N contains no element of the sequence $\langle x_n \rangle$. Finally, let $\mathcal{C}$ denote the collection of open sets $\{N_x \mid x \text{ is an element of } \langle x_n \rangle\} \cup \{N\}$. We see that $\mathcal{C}$ is a countable cover of X that admits no finite subcover. Indeed, omitting *any* N_x results in a subcollection missing x. Therefore, X is not countably compact, thus establishing the contrapositive.

($\Leftarrow$) Let X be a space with the property that every sequence has a cluster point. Take any family of closed sets C_n having the finite intersection property and let x_n be any point belonging to $\bigcap_{i=1}^{n} C_i$. Let x be a cluster point of the sequence $\langle x_n \rangle$ and let O be any open set containing x. Since x is a cluster point, O contains infinitely-many x_n, and hence O intersects every C_n. As O was arbitrary, we conclude that x belongs to every C_n. Hence, $x \in \bigcap_{n=1}^{\infty} C_n$, and so X is countably compact. □

24 Problem 20B

Lemma 24.1. *Let $\mathcal{U}$ be a collection of open sets. For all $U \in \mathcal{U}$,*

$$\mathrm{St}(U,\mathcal{U}) = \bigcup_{x \in U} \mathrm{St}(x,\mathcal{U}).$$

Proof. (To be included if the lemma turns out to be useful) □

Proposition 24.2. *A barycentric refinement of a barycentric refinement of a cover $\mathcal{W}$ is a star-refinement of $\mathcal{W}$.*

Proof. Let $\mathcal{U}\Delta\mathcal{V}\Delta\mathcal{W}$. Our goal is to show that $\mathcal{U} \,^* < \mathcal{W}$. To that end, let $U \in \mathcal{U}$. Now,

$$\begin{aligned}
\mathrm{St}(U,\mathcal{U}) &= \bigcup_{x \in U} \mathrm{St}(x,\mathcal{U}) && \text{(by lemma)} \\
&\subset \bigcup_{x \in U} V_x, \text{ for some } V_x \in \mathcal{V} && \text{(since } \mathcal{U}\Delta\mathcal{V}) \\
&\subset \bigcup_{x \in U} \mathrm{St}(x,\mathcal{V}) \\
&\subset \bigcup_{x \in U} W_x, \text{ for some } W_x \in \mathcal{W} && \text{(since } \mathcal{V}\Delta\mathcal{W}).
\end{aligned}$$

(My problem now is that there is no guarantee that $\bigcup_{x \in U} W_x \in \mathcal{W}$, nor can I force $\bigcup_{x \in U} \mathrm{St}(x,\mathcal{V}) \subset \mathrm{St}(x_0,\mathcal{V})$ for any single x_0.) □

Proposition 24.3. *If $\mathcal{U}_n$ is the cover of a metric space (X, d) by $\left(\frac{1}{3^n}\right)$-balls about each of its points, then $\mathcal{U}_{n+1} \,^* < \mathcal{U}_n$.*

Proof. Let $U \in \mathcal{U}_{n+1}$. By definition, $U = B(x, \frac{1}{3^{n+1}})$ for some $x \in X$. Consider now any point $y \in \mathrm{St}(U,\mathcal{U}_{n+1})$. Now, $y \in V$ for some $V \in \mathcal{U}_{n+1}$ with $U \cap V \neq \emptyset$. It follows that $d(x, y) < \frac{3}{3^{n+1}} = \frac{1}{3^n}$, as x is the center of a $\left(\frac{1}{3^{n+1}}\right)$-ball and V has diameter $\frac{2}{3^{n+1}}$. Hence, $\mathrm{St}(U,\mathcal{U}_{n+1}) \subset B(x, \frac{1}{3^n}) \in \mathcal{U}_n$, as desired. □

Proposition 24.4. *If $\mathcal{U}$ is an open cover of X, $\mathcal{V}$ is an open barycentric refinement of $\mathcal{U}$, and for each $U \in \mathcal{U}$ we define $F_U = X - St(X - U, \mathcal{V})$, then $\{F_U \mid U \in \mathcal{U}\}$ is a closed cover of X.*

Proof. The fact that each F_U is closed is immediate, since it is the complement of a union of open sets. Let now $x \in X$. Since $\mathcal{V}\Delta\mathcal{U}$, $\mathrm{St}(x, \mathcal{V}) \subset U$, for some $U \in \mathcal{U}$. We claim that $x \in F_U$ for this U. Suppose, for the purpose of contradiction, that this is not the case. This means that $x \in \mathrm{St}(X - U, \mathcal{V})$. In particular, there exists some $V \in \mathcal{V}$ with $x \in V$ and $V \cap (X - U) \neq \emptyset$. On the other hand, V is an open set containing x, and so $V \subset \mathrm{St}(x, \mathcal{V}) \subset U$. Having arrived at a contradiction (V cannot be both a subset of U and also intersect $X - U$ nontrivially), we conclude that $x \in F_U$. As x was chosen arbitrarily, we see that $\{F_U \mid U \in \mathcal{U}\}$ is indeed a closed cover of X, as desired. □

Define $\sim$ in any space X by $x \sim y$ if and only if x and y lie together in some connected subset of X. Define $\approx$ in X by $x \approx y$ if and only if there is no decomposition $X = U \cup V$ into disjoint open sets with one containing x and the other containing y.

25 Problem 26B1

Proposition 25.1. *The relation $\sim$ is an equivalence on X. The equivalence class $[x]$ of x is just the component C_x of x in X.*

Proof. The relation $\sim$ is reflexive, as x lies within the connected subset of X containing it (that is, $x \sim x$).

The relation $\sim$ is symmetric, as "x and y lie together in some connected subset" has the same meaning as "y and x lie together in some connected subset".

To see that $\sim$ is transitive, suppose that $x \sim y$ and $y \sim z$ for some $x, y, z \in X$. Let U denote the connected subset of X containing x and y, and let V denote the connected subset of X containing y and z. We see that $U \cap V \neq \emptyset$ (as $y \in U \cap V$), and so $U \cup V$ is connected. Hence, $x \sim z$, as they lie together in the connected subset $U \cup V$ of X.

Taken together, we conclude that $\sim$ is an equivalence relation on X.

Now, let $y \in [x]$. This means that x and y lie in some connected subset of X. Since C_x is the union of *all* connected subsets of X containing x, we see that $y \in C_x$. Next, let $y \in C_x$. Since x and y both lie in the connected component C_x, we see that $y \in [x]$. Hence, $[x] = C_x$, as desired. □

26 Problem 26B2

Proposition 26.1. *The relation $\approx$ is an equivalence on X. We call the equivalence class of x the quasicomponent of x in X. The quasicomponent of x in X is the intersection of all clopen subsets of X that contain x.*

Proof. The relation $\approx$ is reflexive, since there can be no decomposition separating x from itself (that is, $x \sim x$).

The relation $\approx$ is symmetric, as U and V are indistinguishable in the definition. That is, the phrase "one containing x and the other containing y" has the same meaning as "one containing y and the other containing x".

To see that $\approx$ is transitive, we establish the contrapositive. To that end, suppose that $x \not\approx z$. This means that there are disjoint open sets U and V such that (without loss of generality) $x \in U$, $z \in V$, and $X = U \cup V$. Now, it must be that y belongs to one of U and V. If $y \in U$, then U and V represent a decomposition of X of the appropriate type separating y from z, and so $y \not\approx z$. Similarly, if $y \in V$, then U and V separate x from y, and so $x \not\approx y$. Hence, we have that $x \not\approx z \Rightarrow (x \not\approx y) \vee (y \not\approx z)$, thus establishing the contrapositive.

Taken together, we conclude that $\approx$ is an equivalence relation on X.

Let now F denote the intersection of all clopen subsets of X containing x. Let $y \in F$. This means that y belongs to *every* clopen subset of X containing x. In particular, $y \in C_x$. Since C_x is connected, we see that x cannot be separated from y by a pair of disjoint open sets whose union is X (otherwise, these sets would disconnect C_x). Hence, $y \in [x]$. Next, we show that $[x] \subset F$ by establishing the contrapositive. To that end, suppose that $y \notin F$. This means there is some clopen subset F_y of X such that $x \in F_y$ but $y \notin F_y$. It follows that F_y and F_y^c are disjoint open sets with $x \in F_y$, $y \in F_y^c$, and $X = F_y \cup F_y^c$. Hence, $y \notin [x]$, thus establishing the contrapositive. Therefore, $[x] = F$, as desired. □

(I could make no sense of Willard's picture, so I provide a different space where the components and quasicomponents may disagree.)

27 Extra Problem

Proposition 27.1. *Let A denote the set $\{\frac{1}{n} \mid n \in \mathbb{Z}^+\}$ and let X be the space $(A \times [0,1]) \cup \{(0,0)\} \cup \{(0,1)\}$ with the relative Euclidean topology. The points $\{(0,0)\}$ and $\{(0,1)\}$ belong to separate components, but belong to the same quasicomponent.*

Proof. Evidently, $C_{(0,0)} = \{(0,0)\}$, since $\{(0,0)\}$ is the *only* connected subset of X containing $(0,0)$. We claim next that $\{(0,0),(0,1)\} \subset [(0,0)]$. Suppose, for the purpose of contradiction, that there are disjoint open sets U and V such that (without loss of generality) $(0,0) \in U$, $(0,1) \in V$, and $U \cup V = X$. Since U is open, $(0,0) \subsetneq U$. Similarly, $(0,1) \subsetneq V$. Let k be a natural number such that both $U \cap (\frac{1}{k} \times [0,1]) \neq \emptyset$ and $V \cap (\frac{1}{k} \times [0,1]) \neq \emptyset$. Now, as $\frac{1}{k} \times [0,1]$ is a connected subset of X, it must belong entirely to U or V, contradicting our previous assertion that *both* U and V intersect it nontrivially. Therefore, $(0,0)$ and $(0,1)$ belong to separate components, yet they belong to the same quasicomponent. □