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Math 730 Homework

In the following problems, let Λ be an indexing set and let A and B_λ for $\lambda \in \Lambda$ be arbitrary sets.

Problem 1B1

Show $A - \left(\bigcap_{\lambda \in \Lambda} B_\lambda \right) = \bigcup_{\lambda \in \Lambda} (A - B_\lambda)$.

Proof.

$$\begin{aligned} x \in A - \left(\bigcap_{\lambda \in \Lambda} B_\lambda \right) &\Leftrightarrow x \in A \text{ and } x \notin \bigcap_{\lambda \in \Lambda} B_\lambda \\ &\Leftrightarrow x \in A \text{ and } x \notin B_{\lambda_0} \text{ for some } \lambda_0 \in \Lambda \\ &\Leftrightarrow x \in A - B_{\lambda_0} \\ &\Leftrightarrow x \in \bigcup_{\lambda \in \Lambda} (A - B_\lambda) \end{aligned}$$

□

Problem 1H1

Show $f^{\rightarrow} \left(\bigcup_{\lambda \in \Lambda} B_\lambda \right) = \bigcup_{\lambda \in \Lambda} f^{\rightarrow} B_\lambda$.

Proof.

$$\begin{aligned} y \in f^{\rightarrow} \left(\bigcup_{\lambda \in \Lambda} B_\lambda \right) &\Leftrightarrow f(x) = y \text{ for some } x \in \bigcup_{\lambda \in \Lambda} B_\lambda \\ &\Leftrightarrow \exists \lambda_0 \in \Lambda \text{ such that } f(x) = y \text{ with } x \in B_{\lambda_0} \\ &\Leftrightarrow y \in f^{\rightarrow} B_{\lambda_0} \\ &\Leftrightarrow y \in \bigcup_{\lambda \in \Lambda} f^{\rightarrow} B_\lambda \end{aligned}$$

□

Extra Problem

Show that $f : A \to B$ is a bijection if and only if it has a two-sided inverse.

Proof. ($\Rightarrow$) Let f be a bijection. This implies two important facts. Firstly,

$$\begin{aligned} f \text{ bijective} &\Rightarrow f \text{ injective} \\ &\Rightarrow \text{for all } x_0, x_1 \in A,\ x_0 = x_1 \text{ whenever } f(x_0) = f(x_1). \end{aligned}$$

Secondly,

$$\begin{aligned} f \text{ bijective} &\Rightarrow f \text{ surjective} \\ &\Rightarrow \text{for all } y \in A, \text{ there is } x \in A \text{ such that } f(x) = y. \end{aligned}$$

Taken together, we have that

$$f \text{ bijective} \Rightarrow \text{for all } y \in B \text{ there is a unique } x \in A \text{ such that } f(x) = y.$$

In other words, every element of B is of the form $f(x)$ for some unique $x \in A$. Now, define

$$\begin{gathered} g : B \to A \\ g(f(x)) = x \text{ for all } f(x) \in B \end{gathered}$$

We see that, for all $x \in A$,

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= x, \end{aligned}$$

so g is a left inverse of f. We also have, for all $f(x) \in B$,

$$\begin{aligned} (f \circ g)(f(x)) &= f(g(f(x)) \\ &= f(x) \text{ (as } g(f(x)) = x \text{ by definition)}, \end{aligned}$$

and so g is a right inverse of f. Therefore, g is a two-sided inverse.

($\Leftarrow$) Let g be a two-sided inverse of f.

First, suppose that f is not an injection. There are $x_1, x_2 \in A$ such that

$$x_1 \neq x_2 \text{ yet } f(x_1) = f(x_2).$$

It follows that

$$\begin{aligned} x_1 &= g(f(x_1)) \\ &= g(f(x_2)) \text{ (as } f(x_1) = f(x_2)) \\ &= x_2 \end{aligned}$$

which is a contradiction with the fact that $x_1 \neq x_2$. Hence, f is an injection.

Next, suppose that f is not a surjection. There exists $y \in B$ such that

$$f(x) \neq y \text{ for all } x \in A.$$

On the other hand,

$$f(g(y)) = y.$$

Hence, there exists an element of A whose image under f is y (namely $g(y)$), which is a contradiction. Hence, f is a surjection.

Finally, as f is both an injection and a surjection, we conclude that f is indeed a bijection, as desired. □

Problem 2B1 (Metrics on $C(\mathbf{I})$)

Let $C(\mathbf{I})$ denote the set of all continuous, real-valued functions on the unit interval $\mathbf{I}$. Show that

$$\rho(f, g) = \sup_{x \in \mathbf{I}} |f(x) - g(x)|$$

is a metric on $C(\mathbf{I})$.

Proof. We verify each of the three properties of metrics.

Claim 1. $\rho(f, g) \geq 0$ for all $f, g \in C(\mathbf{I})$ with equality if and only if $f = g$.

Proof. By definition, the ouput of absolute value is always nonnegative, so we have that the supremum of a set of absolute values must also be nonnegative. That is, $\rho(f, g) \geq 0$ for all $f, g \in C(\mathbf{I})$. Now,

$$\begin{aligned} \rho(f, g) = 0 \quad &\Leftrightarrow \quad \sup_{x \in \mathbf{I}} |f(x) - g(x)| \\ &\Leftrightarrow \quad |f(x) - g(x)| = 0 \text{ for all } x \in \mathbf{I} \\ &\Leftrightarrow \quad f(x) = g(x) \text{ for all } x \in \mathbf{I} \\ &\Leftrightarrow \quad f = g \text{ on } \mathbf{I} \end{aligned}$$

□

Claim 2. $\rho(f,g) = \rho(g,f)$ for all $f, g \in C(\mathbf{I})$.

Proof. For all $f, g \in C(\mathbf{I})$,

$$\begin{aligned}\rho(f,g) &= \sup_{x\in\mathbf{I}} |f(x) - g(x)| \\ &= \sup_{x\in\mathbf{I}} |g(x) - f(x)| \\ &= \rho(g,f)\end{aligned}$$

□

Claim 3. $\rho(f,g) \leq \rho(f,h) + \rho(h,g)$ for al $f, g, h \in C(\mathbf{I})$.

Proof. For all $f, g, h \in C(\mathbf{I})$,

$$\begin{aligned}\rho(f,g) &= \sup_{x\in\mathbf{I}} |f(x) - g(x)| \\ &\leq \sup_{x\in\mathbf{I}} (|f(x) - h(x)| + |h(x) - g(x)|) \quad (\text{as } |\cdot| \text{ is a metric on } \mathbf{I}) \\ &\leq \sup_{x\in\mathbf{I}} |f(x) - h(x)| + \sup_{x\in\mathbf{I}} |h(x) - g(x)| \\ &= \rho(f,h) + \rho(h,g)\end{aligned}$$

□

Therefore, ρ is a metric on $C(\mathbf{I})$, as desired. □

Lemma 0.1. *Let f be a non-negative, continuous function on $[0,1]$. If $\int_0^1 f(x)\,dx = 0$, then $f = 0$.*

Proof. We prove the claim above by contrapositive. To that end, choose $x_0 \in [0,1]$ such that $f(x_0) = c > 0$. As f is continuous at x_0, there is $\delta > 0$ such that

$$\begin{aligned} |f(x_0) - f(x)| < c \text{ for all } x \in B(x_0,\delta) \quad &\Rightarrow \quad f(x) > f(x_0) - c \text{ for all } x \in B(x_0,\delta) \\ &\Rightarrow \quad f(x) > 0 \text{ for all } x \in B(x_0,\delta). \end{aligned}$$

Now,

$$\begin{aligned} \int_0^1 f(x)\,dx &\geq \int_{B(x_0,\delta)} f(x)\,dx \\ &\geq \delta \cdot \min\{f(x) \mid x \in B(x_0,\delta)\} \\ &> 0 \end{aligned}$$

thus establishing the contrapositive, as desired. □

Problem 2B2 (Metrics on $C(\mathbf{I})$)

Let $C(\mathbf{I})$ denote the set of all continuous, real-valued functions on the unit interval $\mathbf{I}$. Show that

$$\sigma(f,g) = \int_0^1 |f(x) - g(x)| dx$$

is a metric on $C(\mathbf{I})$.

Proof. We verify each of the three properties of metrics.

Claim 4. $\sigma(f,g) \geq 0$ for all $f,g \in C(\mathbf{I})$ with equality if and only if $f = g$.

Proof. By definition, the output of absolute value is always nonnegative, so we have that the integral of the nonnegative function $|f(x) - g(x)|$ is itself nonnegative. That is, $\sigma(f,g) \geq 0$ for all $f,g \in C(\mathbf{I})$. Now, the integral of a nonnegative function is zero if and only if the function is identically zero (call this contention $*$). Hence, we have

$$\begin{aligned} \sigma(f,g) = 0 \quad &\Leftrightarrow \quad \int_0^1 |f(x) - g(x)| dx = 0 \\ &\Leftrightarrow \quad |f(x) - g(x)| = 0 \text{ for all } x \in \mathbf{I} \text{ (by } *\text{)} \\ &\Leftrightarrow \quad f(x) = g(x) \text{ for all } x \in \mathbf{I} \\ &\Leftrightarrow \quad f = g \text{ on } \mathbf{I} \end{aligned}$$

□

Claim 5. $\sigma(f,g) = \sigma(g,f)$ for all $f,g \in C(\mathbf{I})$.

Proof. For all $f,g \in C(\mathbf{I})$,

$$\begin{aligned} \sigma(f,g) &= \int_0^1 |f(x) - g(x)| dx \\ &= \int_0^1 |g(x) - f(x)| dx \\ &= \sigma(g,f) \end{aligned}$$

□

Claim 6. $\sigma(f,g) \leq \sigma(f,h) + \sigma(h,g)$ for al $f,g,h \in C(\mathbf{I})$.

Proof. For all $f, g, h \in C(\mathbf{I})$,

$$\begin{aligned}
\sigma(f,g) &= \int_0^1 |f(x) - g(x)| dx \\
&\leq \int_0^1 (|f(x) - h(x)| + |h(x) - g(x)|) dx \quad \text{(as } |\cdot| \text{ is a metric on } \mathbf{I}\text{)} \\
&= \int_0^1 |f(x) - h(x)| dx + \int_0^1 |h(x) - g(x)| dx \quad \text{(by the linearity of the integral)} \\
&= \sigma(f,h) + \sigma(h,g)
\end{aligned}$$

□

Therefore, σ is a metric on $C(\mathbf{I})$, as desired. □

Problem 2D (Disks are open)

Show that, for any subset A of a metric space (M, d) and any $\epsilon > 0$, the set $B(A, \epsilon)$ is open. (In particular, $B(x, \epsilon)$ is open for each $x \in M$.)

Proof. Let $\epsilon > 0$ be given. Choose $x \in B(A, \epsilon)$. This means that

$$\inf\{d(x, y) \mid y \in A\} < \epsilon$$

This implies that there is some $y_0 \in A$ such that $d(x, y_0) < \epsilon$. Hence, we can find ϵ' such that

$$\epsilon' < \epsilon - d(x, y_0)$$

Claim 7. $B(x, \epsilon') \subset B(A, \epsilon)$

Proof. Let $x' \in B(x, \epsilon')$. We see that

$$\begin{aligned} d(x', A) &= \inf\{d(x', y) \mid y \in A\} \\ &\leq d(x', y_0) \quad (\text{as } y_0 \in A) \\ &\leq d(x', x) + d(x, y_0) \\ &< \epsilon' + d(x, y_0) \\ &< (\epsilon - d(x, y_0)) + d(x, y_0) \\ &= \epsilon \end{aligned}$$

Hence, $x' \in B(A, \epsilon)$, and so $B(x, \epsilon') \subset B(A, \epsilon)$. □

As $B(x, \epsilon')$ contains x and is contained in $B(A, \epsilon)$, we conclude that $B(A, \epsilon)$ is indeed open. As a special case, letting $A = \{x\}$ shows that, for any $x \in M$, $B(x, \epsilon)$ is open. □

Problem 2E1 (Bounded metrics)

A metric ρ on M is bounded if and only if, for some constant A, $\rho(x, y) \leq A$ for all x and y in M. Show that, if ρ is any metric on M, the distance function

$$\rho^*(x, y) = \min\{\rho(x, y), 1\}$$

is a metric and is also bounded.

Proof. We verify each of the three properties of metrics.

Claim 8. $\rho^*(x, y) \geq 0$ for all $x, y \in M$ with equality if and only if $x = y$.

Proof. As ρ is a metric, $\rho(x, y) \geq 0$ for all $x, y \in M$, and hence $\rho^*(x, y) = \min\{\rho(x, y), 1\} \geq 0$ for all $x, y \in M$. Similarly, $\rho(x, y) = 0$ if and only if $x = y$, and so $\rho^*(x, y) \min\{\rho(x, y), 1\} = 0$ if and only if $x = y$. □

Claim 9. $\rho^*(x, y) = \rho^*(x, y)$ for all $x, y \in M$.

Proof. For all $x, y \in M$,

$$\begin{aligned} \rho^*(x, y) &= \min\{\rho(x, y), 1\} \\ &= \min\{\rho(y, x), 1\} \\ &= \rho^*(y, x) \end{aligned}$$

□

Claim 10. $\rho^*(x, y) \le \rho^*(x, z) + \rho^*(z, y)$ for al $x, y, z \in M$.

Proof. For all $x, y, z \in M$,

$$\begin{aligned}
\rho^*(x, y) &= \min\{\rho(x, y), 1\} \\
&\le \min\{\rho(x, z) + \rho(z, y), 1\} \quad \text{(as } \rho \text{ is a metric)} \\
&\le \min\{\rho(x, z), 1\} + \min\{\rho(z, y), 1\} \\
&= \rho^*(x, z) + \rho^*(z, y)
\end{aligned}$$

□

Therefore, ρ^* is a metric on M, as desired. Furthermore, we see that ρ^* is bounded by 1. □

Problem 2E2 (Bounded metrics)
A function f is continuous on (M, ρ) if and only if it is continuous on (M, ρ^*).

Proof. One way to define continuity of f is to say that $f^{\leftarrow}(O)$ is open whenever O is open. Hence, it suffices to show that ρ and ρ^* generate the same collection of open sets (if this holds, then we will have that $f^{\leftarrow}(O)$ and O are open with respect to ρ if and only if they are open with respect to ρ^*). To that end, let $A \subset M$.

$$\begin{aligned}
A \text{ is open with respect to } \rho \quad &\Rightarrow \quad \text{for all } x \in A, \text{ there is } 0 < \epsilon < 1 \text{ such that } B_\rho(x, \epsilon) \subset A \\
&\Rightarrow \quad \rho(x, y) < \epsilon \text{ for all } y \in B_\rho(x, \epsilon) \\
&\Rightarrow \quad \rho^*(x, y) < \epsilon \text{ for all } y \in B_\rho(x, \epsilon) \quad \text{(as } \epsilon < 1\text{)} \\
&\Rightarrow \quad A \text{ is open with respect to } \rho^*
\end{aligned}$$

Similarly,

$$\begin{aligned}
A \text{ is open with respect to } \rho^* \quad &\Rightarrow \quad \text{for all } x \in A, \text{ there is } 0 < \epsilon \text{ such that } B_{\rho^*}(x, \epsilon) \subset A \\
&\Rightarrow \quad \rho^*(x, y) < \epsilon \text{ for all } y \in B_{\rho^*}(x, \epsilon) \\
&\Rightarrow \quad \rho(x, y) < \epsilon \text{ for all } y \in B_{\rho^*}(x, \epsilon) \\
&\Rightarrow \quad A \text{ is open with respect to } \rho
\end{aligned}$$

Therefore, ρ and ρ^* generate the same collection of open sets, and so f is continuous on (M, ρ) if and only if it is continuous on (M, ρ^*). □

Proposition 0.2. *Let (M, ρ) and (N, σ) be pseudometric spaces. If $f : M \to N$ is an isometry, then it is continuous.*

Proof. We aim to establish an alternate phrasing of the notion of continuity, namely that f is continuous at $x \in M$ if, for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$f^{\rightarrow} B_\rho(x, \delta) \subset B_\sigma(f(x), \epsilon).$$

Given $\epsilon > 0$, take $\delta = \epsilon$ and let $x_0 \in B_\rho(x, \epsilon)$. Observe first that $f(x_0) \in f^{\rightarrow} B_\rho(x, \epsilon)$ by definition. Now,

$$\begin{aligned}
x_0 \in B_\rho(x, \epsilon) \quad &\Rightarrow \quad \rho(x, x_0) < \epsilon \\
&\Rightarrow \quad \sigma(f(x), f(x_0)) < \epsilon \qquad (f \text{ is an isometry, so } \sigma(f(x), f(x_0)) = \rho(x, x_0)) \\
&\Rightarrow \quad f(x_0) \in B_\sigma(f(x), \epsilon).
\end{aligned}$$

Thus, we have established that $f^{\rightarrow} B_\rho(x, \epsilon) \subset B_\sigma(f(x), \epsilon)$, and so f is continuous. □

Proposition 0.3. *If $|| \cdot ||$ is an F-pseudonorm on a vectorspace V, then $d(x, y) = ||x - y||$ defines a metric on V.*

Proof. We verify each of the three properties of metrics.

Claim 11. $d(x, y) \ge 0$ for all $x, y \in V$ with equality if and only if $x = y$.

Proof. For all $x, y \in V$, we have that $d(x, y) = ||x - y|| \geq 0$ by definition of F-pseudonorm. Furthermore,

$$\begin{aligned} d(x,y) = 0 \quad &\Leftrightarrow \quad ||x-y|| = 0 \\ &\Leftrightarrow \quad x - y = 0 \qquad \text{(as } ||\cdot|| \text{ is an } F\text{-pseudonorm)} \\ &\Leftrightarrow \quad x = y. \end{aligned}$$

□

Claim 12. $d(x, y) = d(y, x)$ for all $x, y \in V$.

Proof. For all $x, y \in V$,

$$\begin{aligned} d(x,y) &= ||x-y|| \\ &= ||-1(y-x)|| \\ &\leq |-1| \cdot ||y-x|| \qquad \text{(as } ||\cdot|| \text{ is an } F\text{-pseudonorm)} \\ &= ||y-x|| \\ &= d(y,x). \end{aligned}$$

Similarly,

$$\begin{aligned} d(y,x) &= ||y-x|| \\ &= ||-1(x-y)|| \\ &\leq |-1| \cdot ||x-y|| \qquad \text{(as } ||\cdot|| \text{ is an } F\text{-pseudonorm)} \\ &= ||x-y|| \\ &= d(x,y). \end{aligned}$$

Hence, $d(x, y) = d(y, x)$. □

Claim 13. $d(x,z) \le d(x,y) + d(y,z)$ for all $x, y, z \in V$.

Proof. For all $x, y, z \in V$,

$$\begin{aligned} d(x,z) &= ||x - z|| \\ &= ||x + (-1 \cdot z)|| \\ &\le ||x + y|| + ||y + (-1 \cdot z)|| \qquad \text{(as } || \cdot || \text{ is an } F\text{-pseudonorm)} \\ &= ||x + y|| + ||y - z|| \\ &\le ||x + (-1 \cdot y)|| + ||(-1 \cdot y) + y|| + ||y - z|| \qquad \text{(as } || \cdot || \text{ is an } F\text{-pseudonorm)} \\ &= ||x - y|| + ||y - z|| \\ &= d(x,y) + d(y,z). \end{aligned}$$

□

Therefore, d is a metric on V, as desired. □

Proposition 0.4. *(Problem 2C1) Let (M, ρ) be a pseudometric space. The relation $\sim$ defined on M by*

$$x \sim y \text{ if and only if } \rho(x,y) = 0$$

is an equivalence relation.

Proof. We verify each of the three properties of equivalence relations.

Claim 14. $\sim$ is reflexive. That is, for all $x \in M$, $x \sim x$.

Proof. For all $x \in M$, $\rho(x,x) = 0$, as ρ is a pseudometric. Hence, $x \sim x$. □

Claim 15. $\sim$ is symmetric. That is, for all $x, y \in M$, $x \sim y$ implies $y \sim x$.

Proof. For all $x, y \in M$,

$$\begin{aligned} x \sim y \quad &\Rightarrow \quad \rho(x,y) = 0 \\ &\Rightarrow \quad \rho(y,x) = 0 \qquad (\rho \text{ is a pseudometric, and so is symmetric}) \\ &\Rightarrow \quad y \sim x. \end{aligned}$$

□

Claim 16. $\sim$ is transitive. That is, for all $x, y, z \in M$, $x \sim y$ and $y \sim z$ together imply $x \sim z$.

Proof. For all $x, y, z \in M$,

$$\begin{aligned} x \sim y \text{ and } y \sim z \quad &\Rightarrow \quad \rho(x,y) = 0 \text{ and } \rho(y,z) = 0 \\ &\Rightarrow \quad \rho(x,z) = 0 \qquad (\rho \text{ is a pseudometric, so } \rho(x,z) \le \rho(x,y) + \rho(y,z)) \\ &\Rightarrow \quad x \sim z. \end{aligned}$$

□

Therefore, $\sim$ is an equivalence relation on M, as desired. □

Proposition 0.5. *(Problem 2C2) If M^* is the set of equivalence classes in M under the equivalence relation $\sim$ and if ρ^* is defined on M^* by*

$$\rho^*([x],[y]) = \rho(x,y),$$

then ρ^* is a well-defined metric on M^*. (The metric space (M^*, ρ^*) is called the metric identification of (M, ρ).)

Proof.

Claim 17. ρ^* is a well-defined function on M^*.

Proof. Let $x_0, x_1 \in [x] \in M^*$ and $y_0, y_1 \in [y] \in M^*$. As $x_0 \sim x_1$ and $y_0 \sim y_1$, we have

$$\rho(x_0, x_1) = \rho(y_0, y_1) = 0.$$

Now,

$$\begin{aligned}
\rho^*([x_0],[y_0]) &= \rho(x_0, y_0) \\
&\le \rho(x_0, x_1) + \rho(x_1, y_0) \qquad (\rho \text{ is a pseudometric, so we have the triangle inequality}) \\
&\le \rho(x_0, x_1) + \rho(x_1, y_1) + \rho(y_1, y_0) \qquad (\rho \text{ is a pseudometric, so we have the triangle inequality}) \\
&=\le \rho(x_0, x_1) + \rho(x_1, y_1) + \rho(y_0, y_1) \qquad (\rho \text{ is a pseudometric, and so is symmetric}) \\
&= \rho(x_1, y_1) \\
&= \rho^*([x_1],[y_1]).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\rho^*([x_1],[y_1]) &= \rho(x_1, y_1) \\
&\le \rho(x_1, x_0) + \rho(x_0, y_1) \qquad (\rho \text{ is a pseudometric, so we have the triangle inequality}) \\
&\le \rho(x_1, x_0) + \rho(x_0, y_0) + \rho(y_0, y_1) \qquad (\rho \text{ is a pseudometric, so we have the triangle inequality}) \\
&=\le \rho(x_0, x_1) + \rho(x_0, y_0) + \rho(y_0, y_1) \qquad (\rho \text{ is a pseudometric, and so is symmetric}) \\
&= \rho(x_0, y_0) \\
&= \rho^*([x_0],[y_0]).
\end{aligned}$$

Hence, $\rho^*([x_0],[y_0]) = \rho^*([x_1],[y_1])$, and so ρ^* is a well-defined function on M^*. □

We proceed by verifying each of the three properties of metrics for ρ^*.

Claim 18. $\rho^*([x],[y]) \ge 0$ for all $[x],[y] \in M^*$ with equality if and only if $[x] = [y]$.

Proof. As ρ is a metric, it is non-negative, and so ρ^* is non-negative. Now, for all $[x],[y] \in M^*$,

$$\begin{aligned}
\rho^*([x],[y]) = 0 \quad &\Leftrightarrow \quad \rho(x, y) = 0 \\
&\Leftrightarrow \quad x \sim y \\
&\Leftrightarrow \quad [x] = [y] \qquad (\text{equivalence classes are either disjoint or they coincide}).
\end{aligned}$$

□

Claim 19. $\rho^*([x],[y]) = \rho([y],[x])$ for all $[x],[y] \in M^*$.

Proof. For all $[x],[y] \in M^*$,

$$\begin{aligned}
\rho^*([x],[y]) &= \rho(x, y) \\
&= \rho(y, x) \qquad (\rho \text{ is a pseudometric, and so is symmetric}) \\
&= \rho^a st([y],[x])
\end{aligned}$$

□

Claim 20. $\rho^*([x],[z]) \le \rho^*([x],[y]) + \rho^*([y],[z])$ for all $[x],[y],[z] \in M^*$.

Proof. For all $[x],[y],[z] \in M^*$,

$$\begin{aligned}
\rho^*([x],[z]) &= \rho(x, z) \\
&\le \rho(x, y) + \rho(y, z) \qquad (\rho \text{ is a metric, so we have the triangle inequality}) \\
&= \rho^*([x],[y]) + \rho^*([y],[z])
\end{aligned}$$

□

Therefore, ρ^* is a well-defined metric on M^*, as desired. □

Definition 0.6. A normed linear space is a real linear space X such that a number $||x||$, the norm of x, is associated with each $x \in X$, satisfying

(i) $||x|| \geq 0$, and $||x|| = 0$ if and only if $x = 0$;

(ii) $||\alpha x|| = |a| \cdot ||x||$, for all $\alpha \in \mathbb{R}$;

(iii) $||x + y|| \leq ||x|| + ||y||$.

If (i) is replaced by the weaker condition

(i^-) $||x|| \geq 0$ and $||0|| = 0$,

then X is a pseudonormed linear space.

Proposition 0.7. *(Problem 2J2) If $||\cdot||_1$ and $||\cdot||_2$ are pseudonorms on the same linear space X, they give the same open sets (i.e. are equivalent) if and only if there are constants C and C' such that $||x||_1 \leq C \cdot ||x||_2$ and $||x||_2 \leq C' \cdot ||x||_1$, for all $x \in X$.*

Proof. ($\Rightarrow$) For any $x \in X$, define

$$B_1(\vec{0}, r) = \{x \in X \mid ||x||_1 < r\}$$
$$B_2(\vec{0}, r) = \{x \in X \mid ||x||_2 < r\}$$

Now, as $B_1(\vec{0}, r)$ is open with respect to $||\cdot||_1$, it is open with respect to $||\cdot||_2$ by hypothesis. Hence, there is $\epsilon_2 > 0$ such that $B_2(\vec{0}, \epsilon_2) \subset B_1(\vec{0}, r)$. (Simiarly, we can find $\epsilon_1 > 0$ such that $B_1(\vec{0}, \epsilon_1) \subset B_2(\vec{0}, r)$.) (I fail to see the next step. The chosen ϵ_i give some open ball contained in the larger ball, but there is no guarantee that any x in, say, $B_1(\vec{0}, r)$ can be found in $B_2(\vec{0}, \epsilon_2)$, so I cannot make any claim about $||x||_2$.)

($\Leftarrow$) Let U be open with respect to $||\cdot||_1$. That is, for all $x \in U$, there exists $\epsilon > 0$ such that

$$B_1(x, \epsilon) \subset U,$$

where the subscript "1" denotes that distance is computed using $||\cdot||$. Now,

$$\begin{aligned}
y \in B_2(x, \frac{\epsilon}{C}) &\Rightarrow ||x - y||_2 < \frac{\epsilon}{C} \\
&\Rightarrow ||x - y||_1 < C \cdot \frac{\epsilon}{C} \qquad \text{(since } ||x - y||_1 \leq C||x - y||_2\text{)} \\
&\Rightarrow ||x - y||_1 < \epsilon \\
&\Rightarrow y \in B_1(x, \epsilon) \\
&\Rightarrow B_2(x, \frac{\epsilon}{C}) \subset B_1(x, \epsilon)
\end{aligned}$$

Hence, U is open with respect to $||\cdot||_2$. Similarly, all sets open with respect to $||\cdot||_2$ are open with respect to $||\cdot||_1$, and so $||\cdot||_1$ and $||\cdot||_2$ are equivalent. □

1 Extra Problem 1

Lemma 1.1. *Let $\hat{}$ be a Čech closure operation. If $A \subset B$, then $\hat{A} \subset \hat{B}$.*

Proof. It follows directly from the properties of Čech closure operations that

$$\begin{aligned}
A \subset B &\Rightarrow A \cup B = B \\
&\Rightarrow \widehat{A \cup B} = \hat{B} \\
&\Rightarrow \hat{A} \cup \hat{B} = \hat{B}.
\end{aligned}$$

Now, $\hat{A} \subset \hat{A} \cup \hat{B} = \hat{B}$, and so $\hat{A} \subset \hat{B}$, as desired. □

Proposition 1.2. *If we define a set in a Čech closure space $(X, \hat{\ })$ to be closed if $A = \hat{A}$, then the result is a topology.*

Proof. Let $\mathcal{F}$ denote the collection of subsets of X that are closed with respect to $\hat{\ }$. By Homework 4, Problem 2, if $\mathcal{F}$ satisfies

(F-a) the intersection of an arbitrary collection of elements of $\mathcal{F}$ belongs to $\mathcal{F}$,

(F-b) the union of a finite collection of elements of $\mathcal{F}$ belongs to $\mathcal{F}$, and

(F-c) the sets $\emptyset$ and X belong to $\mathcal{F}$,

then $\tau = \{F^c \mid F \in \mathcal{F}\}$ is a topology on X. We verify that $\mathcal{F}$ indeed satisfies each of these three properties.

Claim 21. The intersection of an arbitrary collection of elements of $\mathcal{F}$ belongs to $\mathcal{F}$.

Proof. Let $F_\alpha \in \mathcal{F}$ for all α belonging to some indexing set I. As $\hat{\ }$ is a Čech closure operation,

$$\bigcap_{\alpha \in I} F_\alpha \subset \widehat{\bigcap_{\alpha \in I} F_\alpha}.$$

To see the reverse inclusion, observe that

$$\begin{aligned} & \bigcap_{\alpha \in I} F_\alpha \subset F_\alpha \text{ for each } \alpha \in I \\ \Rightarrow\ & \widehat{\bigcap_{\alpha \in I} F_\alpha} \subset \hat{F}_\alpha \text{ for each } \alpha \in I \quad \text{(by 1.1)} \\ \Rightarrow\ & \widehat{\bigcap_{\alpha \in I} F_\alpha} \subset \bigcap_{\alpha \in I} \hat{F}_\alpha \\ \Rightarrow\ & \widehat{\bigcap_{\alpha \in I} F_\alpha} \subset \bigcap_{\alpha \in I} F_\alpha \qquad \text{(as each } F_\alpha \text{ is closed).} \end{aligned}$$

Hence, $\bigcap_{\alpha \in I} F_\alpha = \widehat{\bigcap_{\alpha \in I} F_\alpha}$ (i.e. $\bigcap_{\alpha \in I} F_\alpha$ is closed in $\mathcal{F}$). ☐

Claim 22. The union of a finite collection of elements of $\mathcal{F}$ belongs to $\mathcal{F}$.

Proof. Let $F_i \in \mathcal{F}$ for $1 \leq i \leq n$. We have that

$$\begin{aligned} \bigcup_{i=1}^{n} F_i &= \bigcup_{i=1}^{n} \hat{F}_i \qquad \text{(since } F_i = \hat{F}_i \text{ for all } i\text{)} \\ &= \widehat{\bigcup_{i=1}^{n} F_i} \qquad \text{(by induction on } i\text{, using the fact that } \widehat{A \cup B} = \hat{A} \cup \hat{B} \text{ for the Čech closure operation } \hat{\ }\text{)} \end{aligned}$$

☐

Claim 23. The sets $\emptyset$ and X belong to $\mathcal{F}$.

Proof. We have that $\emptyset = \hat{\emptyset}$ by definition of Čech closure operation, so $\emptyset \in \mathcal{F}$. Now,

$$\begin{aligned} X &\subset \hat{X} \qquad \text{(since } \hat{\ } \text{ is a Čech closure operation)} \\ &\subset X \qquad \text{(since } \hat{\ } : \mathcal{P}(X) \to \mathcal{P}(X)\text{).} \end{aligned}$$

Hence, $X = \hat{X}$, and so $X \in \mathcal{F}$. ☐

Therefore, $\tau = \{F^c \mid F \in \mathcal{F}\}$ is a topology on X, as desired. ☐

2 Problem 3A1

Proposition 2.1. *If $\mathcal{F}$ is the collection of all closed, bounded subsets of $\mathbb{R}$ (in its usual topology), together with $\mathbb{R}$ itself, then $\mathcal{F}$ is the family of closed sets for a topology on $\mathbb{R}$ strictly weaker than the usual topology.*

Proof. We proceed by verifying properties F-a, F-b, and F-c on the family of sets $\mathcal{F}$.

Claim 24. The intersection of an arbitrary collection of elements of $\mathcal{F}$ belongs to $\mathcal{F}$.

Proof. Let $F_\alpha \in \mathcal{F}$ for all α belonging to some indexing set I. In $\mathbb{R}$, the intersection of an arbitrary collection of closed sets is closed. Hence, $\bigcap_{\alpha \in I} F_\alpha$ is closed. Furthermore, each F_α is bounded, and so $\bigcap_{\alpha \in I} F_\alpha$ (which is contained in F_α for every α) is also bounded. Thus, $\bigcap_{\alpha \in I} F_\alpha$ is closed and bounded, and so belongs to $\mathcal{F}$. □

Claim 25. The union of a finite collection of elements of $\mathcal{F}$ belongs to $\mathcal{F}$.

Proof. Let $F_i \in F$ for all $1 \leq in$. In $\mathbb{R}$, the union of a finite collection of closed sets is closed. Hence, $\bigcup_{i=1}^{n} F_i$ is closed. Now, each F_i is bounded by some $B(0, \epsilon_i)$. Take $\epsilon = \max\{\epsilon_i \mid 1 \leq i \leq n\}$. We see that $F_i \subset B(0, \epsilon)$ for all $1 \leq i \leq n$, and so $\bigcup_{i=1}^{n} F_i \subset B(0, \epsilon)$. Thus, $\bigcup_{i=1}^{n} F_i$ is closed and bounded, and so belongs to $\mathcal{F}$. □

Claim 26. The sets $\emptyset$ and $\mathbb{R}$ belong to $\mathcal{F}$.

Proof. We have that $\mathbb{R} \in \mathcal{F}$ by definition. Now, $\emptyset$ is trivially closed and $\emptyset \subset B(0, 1)$. Hence, $\emptyset$ is closed and bounded, and so belongs to $\mathcal{F}$. □

Therefore, $\mathcal{F}$ is the family of closed subsets for the topology $\tau = \{F^c \mid F \in \mathcal{F}\}$. To see that τ is weaker than the usual Euclidean topology (call it τ'), observe that $B(0, 1) \in \tau'$ (since open balls are open), but $B(0, 1) \notin \tau$ (since $B(0, 1)^c = \mathbb{R} \setminus B(0, 1)$ is unbounded). □

3 Problem 3E2

Proposition 3.1. *Let X be a metrizable space whose topology is generated by a metric ρ. The closure of a set $E \subset X$ is given by*

$$\overline{E} = \{y \in X \mid \rho(E, y) = 0\}.$$

Proof. ($\subset$) Let $x \in \overline{E}$. It follows that,

$$\begin{aligned} x \in \overline{E} &\to G \cap E \neq \emptyset \text{ for all open } G \text{ containing } x \\ &\to B(x, \epsilon) \cap E \neq \emptyset \text{ for all } \epsilon > 0 \\ &\to d(E, x) < \epsilon \text{ for all } \epsilon > 0 \\ &\to d(E, x) = 0 \\ &\to x \in \{y \in X \mid \rho(E, y) = 0\}. \end{aligned}$$

Hence, $\overline{E} \subset \{y \in X \mid \rho(E, y) = 0\}$.

($\supset$) Let $x \in \{y \in X \mid \rho(E, y) = 0\}$. It follows that,

$$\begin{aligned} x \in \{y \in X \mid \rho(E, y) = 0\} &\to \rho(E, x) = 0 \\ &\to B(x, \epsilon) \cap E \neq \emptyset \text{ for all } \epsilon > 0. \end{aligned}$$

Now, for any open G containing x, there must exist $\epsilon_0 > 0$ such that $B(x, \epsilon_0) \subset G$ (by definition of openness in a metrizable space). Hence,

$$\begin{aligned} B(x,\epsilon) \cap E \neq \emptyset \text{ for all } \epsilon > 0 &\to G \cap E \neq \emptyset \text{ for all open } G \text{ containing } x \\ &\to x \in \overline{E}. \end{aligned}$$

Hence, $\{y \in X \mid \rho(E, y) = 0\} \subset \overline{E}$. □

4 Problem 3E3

Proposition 4.1. *Let X be a metrizable space whose topology is generated by a metric ρ. The closed disk $U(x, \overline{\epsilon}) = \{y \in X \mid \rho(x, y) \leq \epsilon\}$ is closed in X, but may not be the closure of the open disk $U(x, \epsilon)$.*

Proof. We show, equivalently, that $U(x, \overline{\epsilon})^c$ is open. As X is a metrizable space, we need to establish that, for any $z \in U(x, \overline{\epsilon})^c$, there exists $\epsilon' > 0$ such that $B(z, \epsilon') \subset U(x, \overline{\epsilon})^c$. To that end, let $\epsilon' = \rho(x, z) - \epsilon$ (which, a priori, may or may not be positive). Now,

$$\begin{aligned} B(z, \epsilon') \subset U(x, \overline{\epsilon})^c &\Leftrightarrow B(z, \epsilon') \subset \{y \in X \mid \rho(x, y) > \epsilon\} \\ &\Leftrightarrow \rho(x, w) > \epsilon \text{ for all } w \in B(z, \epsilon'). \end{aligned}$$

Applying the triangle inequality, we have

$$\begin{aligned} \rho(x, z) &\leq \rho(x, w) + \rho(w, z) \\ &< \rho(x, w) + (\rho(x, z) - \epsilon) \qquad \text{(note now that } \rho(x, z) > \epsilon\text{, so } \rho(x, z) - \epsilon > 0). \end{aligned}$$

Rearranging the terms yeilds that $\rho(x, w) > \epsilon$, as desired.

As an example when $U(x, \overline{\epsilon}) \neq \overline{U(x, \epsilon)}$, let $X = \mathbb{R}$ and ρ be the discrete metric. Observe that $U(0, \overline{1}) = \mathbb{R}$, but

$$\begin{aligned} \overline{U(0,1)} &= \bigcap \{K \mid K \text{ is closed and contains } U(0,1)\} \\ &= \bigcap \{K \mid K \text{ is closed and contains } \{0\}\} \\ &= \{0\} \qquad \text{(since } \{0\} \text{ itself is closed in a metric space).} \end{aligned}$$

□

5 Extra Problem 1

Proposition 5.1. *A set in a pseudometric space is open if and only if it is a union of open disks.*

Proof. ($\Rightarrow$) Let O be an open set in a pseudometric space. To be open means that, for each $x \in O$, there is $\epsilon_x > 0$ such that $B(x, \epsilon_x) \subset O$. We claim that

$$O = \bigcup_{x \in O} B(x, \epsilon_x).$$

To see this, choose $x_0 \in O$. By definition of openness, $x_0 \in B(x_0, \epsilon_{x_0})$, but $B(x_0, \epsilon_{x_0}) \subset \bigcup_{x \in O} B(x, \epsilon_x)$, so $x_0 \in \bigcup_{x \in O} B(x, \epsilon_x)$.

To get the reverse inclusion, choose $x_0 \in \bigcup_{x \in O} B(x, \epsilon_x)$. By construction, $x_0 \in B(x_0, \epsilon_{x_0})$. Now, ϵ_{x_0} was chosen specifically to ensure that $B(x_0, \epsilon_{x_0}) \subset O$. Hence, $x \in O$.

Therefore, $O = \bigcup_{x \in O} B(x, \epsilon_x)$. That is, O is the union of open disks.

(⇐) Let $\bigcup_{\alpha \in I} B_\alpha$ be an arbitrary union of open disks.

$$x \in \bigcup_{\alpha \in I} B_\alpha \quad \Rightarrow \quad x \in B_{\alpha_0} \text{ for some } \alpha_0 \in I$$

As open disks are open, we have successfully found an open set contained in $\bigcup_{\alpha \in I} B_\alpha$ that contains x. Therefore, $\bigcup_{\alpha \in I} B_\alpha$ is open. □

6 Extra Problem 2

Proposition 6.1. *Let $\mathcal{F}$ be any collection of subsets of a set X. If $\mathcal{F}$ satisfies*

(i) $\bigcap_{\alpha \in I} F_\alpha \in \mathcal{F}$ *if* $F_\alpha \in \mathcal{F}$ *for all* α *belonging to some indexing set* I

(ii) $\bigcup_{i=1}^{n} F_i \in \mathcal{F}$ *if* $F_i \in \mathcal{F}$ *for all* i

(iii) $\emptyset \in \mathcal{F}$ *and* $X \in \mathcal{F}$,

then $\tau = \{F^c \mid F \in \mathcal{F}\}$ *is a topology on* X *and* $\mathcal{F}$ *is the collection of closed sets in this topology.*

Proof. We show that τ satisfies each of the three properties of topologies on X.

Claim 27. $\bigcup_{\alpha \in I} F_\alpha^c \in \tau$ if $F_\alpha^c \in \tau$ for all α belonging to some indexing set I.

Proof. By problem 1B1, we have

$$\bigcup_{\alpha \in I} F_\alpha^c = \left(\bigcap_{\alpha \in I} F_\alpha \right)^c.$$

By hypothesis, $\bigcap_{\alpha \in I} F_\alpha$ is closed, and so its complement is open. Hence, $\bigcup_{\alpha \in I} F_\alpha^c \in \tau$. □

Claim 28. $\bigcap_{i=1}^{n} F_i^c \in \tau$ if $F_i^c \in \tau$ for all i.

Proof. As a corollary of problem 1B1 (take the complement of both sides), we have

$$\bigcap_{i=1}^{n} F_i^c = \left(\bigcup_{i=1}^{n} F_i \right)^c.$$

By hypothesis, $\bigcup_{i=1}^{n} F_i$ is closed, and so its complement is open. Hence, $\bigcap_{i=1}^{n} F_i^c \in \tau$. □

Claim 29. $\emptyset \in \tau$ and $X \in \tau$.

Proof. We have that $\emptyset = X^c$ and $X \in \mathcal{F}$, so $\emptyset \in \tau$. Also, $X = \emptyset^c$ and $\emptyset \in \mathcal{F}$, so $X \in \tau$. □

Therefore, τ is a topology on X.

It remains to show that $\mathcal{F}$ is the collection of closed sets in τ. We have immediately that any element $F \in \mathcal{F}$ is closed (since $F^c \in \tau$, and so open). Similarly, if a subset K of X is closed, then K^c is open. Thus, $K^c \in \tau$, which implies that $K \in \mathcal{F}$ (τ is precisely the collection of complements of sets from $\mathcal{F}$). Therefore, the collection of closed sets in τ and the collection $\mathcal{F}$ conincide. □

7 Problem 2F2

Definition 7.1. Let ρ be a bounded metric on M; that is, for some constant A,

$$\rho(x,y) \leq A \text{ for all } x,y \in M.$$

Let $\mathcal{F}(M)$ be all nonempty closed subsets of M and for $A,B \in \mathcal{F}(M)$ define

$$\begin{aligned} d_A(B) &= \sup\{\rho(A,x) \mid x \in B\} \\ d(A,B) &= \max\{d_A(B), d_B(A)\}. \end{aligned}$$

Lemma 7.2. *For all $A,B,C \in \mathcal{F}(M)$,*

$$\sup\{\rho(A,c) \mid c \in C\} \leq \sup\{\rho(A,b) + \rho(B,c) \mid b \in B, c \in C\}.$$

Proof. For all $A,B,C \in \mathcal{F}(M)$,

$$\begin{aligned} \sup\{\rho(A,c) \mid c \in C\} &= \sup\{\inf\{\rho(a,c) \mid a \in A\} \mid c \in C\} \\ &\leq \sup\{\inf\{\rho(a,b) + \rho(B,c) \mid a \in A, b \in B\} \mid c \in C\} \\ &= \sup\{\inf\{\rho(a,b) \mid a \in A, b \in B\} + \rho(B,c) \mid c \in C\} \\ &\leq \sup\{\inf\{\rho(a,b) \mid a \in A\} + \rho(B,c) \mid b \in B, c \in C\} \\ &= \sup\{\rho(A,b) + \rho(B,c) \mid b \in B, c \in C\}. \end{aligned}$$

□

Proposition 7.3. *The function d is a metric on $\mathcal{F}(M)$ with the property that $d(\{x\},\{y\}) = \rho(x,y)$ (called the Hausdorff metric on $\mathcal{F}(M)$).*

Proof. We show that d satisfies each of the three properties of metrics on $\mathcal{F}(M)$.

Claim 30. For all $A,B \in \mathcal{F}(M)$, $d(A,B) \geq 0$ with equality if and only if $A = B$.

Proof. As ρ is a metric on M,

$$\begin{aligned} \rho(x,y) \geq 0 \text{ for all } x,y \in M \quad &\Rightarrow \quad d_A(B) \geq 0 \text{ for all } A,B \in \mathcal{F}(M) \\ &\Rightarrow \quad d(A,B) \geq 0 \text{ for all } A,B \in \mathcal{F}(M). \end{aligned}$$

Now,

$$\begin{aligned} d(A,B) = 0 \quad &\Leftrightarrow \quad d_A(B) = 0 \text{ and } d_B(A) = 0 \\ &\Leftrightarrow \quad \sup\{\rho(A,b) \mid b \in B\} = 0 \text{ and } \sup\{\rho(B,a) \mid a \in A\} = 0 \\ &\Leftrightarrow \quad \rho(A,b) = 0 \text{ for all } b \in B \text{ and } \rho(B,a) = 0 \text{ for all } a \in A \\ &\Leftrightarrow \quad A \subset B \text{ and } B \subset A \qquad \text{(since } A \text{ and } B \text{ are closed)} \\ &\Leftrightarrow \quad A = B. \end{aligned}$$

□

Claim 31. For all $A,B \in \mathcal{F}(M)$, $d(A,B) = d(B,A)$.

Proof. For all $A,B \in \mathcal{F}(M)$,

$$\begin{aligned} d(A,B) &= \max\{d_A(B), d_B(A)\} \\ &= \max\{d_B(A), d_A(B)\} \\ &= d(B,A). \end{aligned}$$

□

Claim 32. For all $A, B, C \in \mathcal{F}(M)$, $d(A, C) \leq d(A, B) + d(B, C)$.

Proof. Let $A, B, C \in \mathcal{F}(M)$. Without loss of generality, assume A and C are such that

$$\begin{aligned} d(A, C) &= \max\{d_A(C), d_C(A)\} \\ &= d_A(C). \end{aligned}$$

Now,

$$\begin{aligned} d(A, C) &= d_A(C) \\ &= \sup\{\rho(A, c) \mid c \in C\} \\ &\leq \sup\{\rho(A, b) + \rho(B, c) \mid b \in B, c \in C\} \qquad \text{(by 7.2)} \\ &= \sup\{\rho(A, b) \mid b \in B\} + \sup\{\rho(B, c) \mid c \in C\} \\ &= d_A(B) + d_B(C) \\ &\leq \max\{d_A(B),\ d_B(A)\} + \max\{d_B(C),\ d_C(B)\} \\ &= d(A, B) + d(B, C). \end{aligned}$$

□

Therefore, d is a metric on $\mathcal{F}(M)$.

Moreover, for all $x, y \in M$,

$$\begin{aligned} d(\{x\}, \{y\}) &= \max\{d_{\{x\}}(\{y\}), d_{\{y\}}(\{x\})\} \\ &= \max\{\sup\{\rho(\{x\}, z) \mid z \in \{y\}\}, \sup\{\rho(\{y\}, z) \mid z \in \{x\}\}\} \\ &= \max\{\sup\{\rho(x, y)\}, \sup\{\rho(y, x)\}\} \\ &= \max\{\rho(x, y), \rho(y, x)\} \\ &= \max\{\rho(x, y), \rho(x, y)\} \\ &= \rho(x, y). \end{aligned}$$

□

8 Problem 3A2

Proposition 8.1. *If $A \subset X$, then the family τ of all subsets of X which contain A, together with the empty set ϕ, is a topology on X.*

Proof. We show that τ satisfies each of the three properties of topologies on X.

Claim 33. If $O_\alpha \in \tau$ for all α belonging to some indexing set I, then $\bigcup_{\alpha \in I} O_\alpha \in \tau$.

Proof. Let $O_\alpha \in \tau$ for all $\alpha \in I$. If this collection consists of *only* the empty set, then the union is itself empty, and so the union belongs to τ. Otherwise, we may disregard any occurrence of the empty set in the collection (it contributes nothing to the union). Adopting this convention, we have

$$\begin{aligned} A \subset O_\alpha \text{ for all } \alpha \in I &\Rightarrow A \subset \bigcup_{\alpha \in I} O_\alpha \\ &\Rightarrow \bigcup_{\alpha \in I} O_\alpha \in \tau. \end{aligned}$$

□

Claim 34. If $O_i \in \tau$ for all $1 \leq i \leq n$, then $\bigcap_{i=1}^{n} O_i \in \tau$.

Proof. Let $O_i \in \tau$ for $1 \leq i \leq n$. If O_k is the empty set for any k, then the intersection is itself empty, and so the intersection belongs to τ. Otherwise,

$$\begin{aligned} A \subset O_i \text{ for } 1 \leq i \leq n \Rightarrow A &\subset \bigcap_{i=1}^{n} O_i \\ &\Rightarrow \bigcap_{i=1}^{n} O_i \in \tau. \end{aligned}$$

□

Claim 35. The empty set belongs to τ and the set X belongs to τ.

Proof. The empty set is included in τ by definition, and it is clear that $A \subset X$, so $X \in tau$. □

Therefore, τ is a topology on X. □

Remark 8.2. Recall that the interior of a subset B of X is defined as

$$B^\circ = \bigcup \{O \mid O \text{ open}, O \subset B\}.$$

Now, if $A \not\subset B$, then the only open subset contained in B is $\emptyset$, and so $B^\circ = \emptyset$. Otherwise, B is itself an open set, and so $B^\circ = B$.

Recall that the closure of a subset B of X is defined as

$$\overline{B} = \bigcap \{F \mid F \text{ closed}, B \subset F\}.$$

Observe that

$$\begin{aligned} F \text{ is closed} &\Leftrightarrow F^c \text{ is open} \\ &\Leftrightarrow A \subset F^c \\ &\Leftrightarrow A \cap F = \emptyset. \end{aligned}$$

Now, if $A \cap B \neq \emptyset$, then the only closed subset containing B is X, and so $\overline{B} = X$. Otherwise, B is itself a closed set, and so $\overline{B} = B$.

Remark 8.3. If $A = \phi$, then every subset of X is open (as every set contains $\emptyset$), so τ is the discrete topology. If $A = X$, then only X and $\emptyset$ are open (as X can only be contained in itself and we define $\emptyset$ to be open), so τ is the indiscrete topology.

9 Problem 3C

Proposition 9.1. *If A is any subset of a topological space, the largest possible number of sets in the two sequences*

$$\begin{gathered} A, A^-, A^{-c}, A^{-c-}, \cdots \\ A, A^c, A^{c-}, A^{c-c}, \cdots \end{gathered}$$

(where ^{c} denotes complementation and $^-$ denotes closure) is 14. Furthermore, there is a subset of $\mathbb{R}$ that gives 14.

Proof. Let A be any subset of a topological space and denote the interior operation by $^\circ$. In the first sequence, we see that

$$\begin{aligned}
A^{-c-c-c-} &= (A^{-c})^{-c-c-} \\
&= (A^{c\circ})^{-c-c-} && \text{(since } E^{-c} = E^{c\circ} \text{ for all sets } E) \\
&= (A^{c\circ-})^{c-c-} \\
&= (A^{c\circ-})^{\circ cc-} && \text{(since } E^{c-} = E^{\circ c} \text{ for all sets } E) \\
&= A^{c\circ-\circ cc-} \\
&= A^{c\circ-\circ-} && \text{(since } E^{cc} = E \text{ for all sets } E) \\
&= (A^{c\circ})^{-\circ-} \\
&= (A^{c\circ})^{-} && \text{(since } A^{c\circ} \text{ is open and } G^{-\circ-} = G^{-} \text{ for all } \textit{open} \text{ sets } G) \\
&= (A^{-c})^{-} && \text{(since } E^{c\circ} = E^{-c} \text{ for all sets } E) \\
&= A^{-c-},
\end{aligned}$$

which already appears on the list. Hence, we can get at most seven sets in this way (including the original set A).

In the second sequence, we have

$$\begin{aligned}
A^{c-c-c-c-} &= (A^{c})^{-c-c-c-} \\
&= (A^{c})^{-c-} && \text{(by the previous argument)},
\end{aligned}$$

which already appears on the list. Hence, we can get at most seven new sets in this way (here, we exclude the original set A, as it has alerady been counted). In total, then, there can be at most 14 distinct sets in these sequences.

We exhibit a subset of $\mathbb{R}$ that achieves the bound. Let

$$A = [0,1] \cup (2,3) \cup \{(4,5) \cap \mathbb{Q}\} \cup \{(6,8) - \{7\}\} \cup \{9\}.$$

The first sequence gives

$$\begin{aligned}
A^{-} &= [0,1] \cup [2,3] \cup [4,5] \cup [6,8] \cup \{9\} \\
A^{-c} &= (-\infty,0) \cup (1,2) \cup (3,4) \cup (5,6) \cup (8,9) \cup (9,\infty) \\
A^{-c-} &= (-\infty,0] \cup [1,2] \cup [3,4] \cup [5,6] \cup [8,\infty) \\
A^{-c-c} &= (0,1) \cup (2,3) \cup (4,5) \cup (6,8) \\
A^{-c-c-} &= [0,1] \cup [2,3] \cup [4,5] \cup [6,8] \\
A^{-c-c-c} &= (-\infty,0) \cup (1,2) \cup (3,4) \cup (5,6) \cup (8,\infty),
\end{aligned}$$

and the second sequence gives

$$\begin{aligned}
A^{c} &= (-\infty,0) \cup (1,2] \cup [3,4] \cup \{(4,5) - \mathbb{Q}\} \cup [5,6] \cup \{7\} \cup [8,9) \cup (9,\infty) \\
A^{c-} &= (-\infty,0] \cup [1,2] \cup [3,6] \cup \{7\} \cup [8,\infty) \\
A^{c-c} &= (0,1) \cup (2,3) \cup (6,7) \cup (7,8) \\
A^{c-c-} &= [0,1] \cup [2,3] \cup [6,8] \\
A^{c-c-c} &= (-\infty,0) \cup (1,2) \cup (3,6) \cup (8,\infty) \\
A^{c-c-c-} &= (-\infty,0] \cup [1,2] \cup [3,6] \cup [8,\infty) \\
A^{c-c-c-c} &= (0,1) \cup (2,3) \cup (6,8),
\end{aligned}$$

giving a total of 14 distinct sets, thus meeting the upper bound. □

10 Problem 4A3

Definition 10.1. The Sorgenfrey line, denoted $\mathbf{E}$, is the real line with the topology in which basic neighborhoods of x are the sets $[x, z)$ for $z > x$.

Proposition 10.2. *The closure in* $\boldsymbol{E}$ *of* $\mathbb{Q}$ *is* $\mathbb{R}$.

Proposition 10.3. *The closure in* $\boldsymbol{E}$ *of* $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ *is itself together with* $\{0\}$.

Proposition 10.4. *The closure in* $\boldsymbol{E}$ *of* $\{-\frac{1}{n} \mid n \in \mathbb{N}\}$ *is itself.*

Proposition 10.5. *The closure in* $\boldsymbol{E}$ *of the integers is itself.*

11 Problem 4B1

Definition 11.1. Let Γ denote the closed upper half-plane $\{(x, y) \mid y \geq 0\}$ in $\mathbb{R}^2$. For each point in the open upper half-plane, basic neighborhoods will be the usual open disks (with the restriction, of course, that they be taken small enough to lie in Γ). At the points z on the x-axis, the basic neighborhoods will be the sets $\{z\} \cup A$, where A is an open disk in the upper half-plane, tangent to the x-axis at z. This collection of basic neighborhoods is known as the Moore plane.

Proposition 11.2. *The Moore plane gives a topology on* Γ.

Proof. Recall that, if a collection $\mathcal{B}_x$ of subsets of X is assigned to each $x \in X$ so as to satisfy

V-a) if $V \in \mathcal{B}_x$, then $x \in V$,

V-b) if $V_1, V_2 \in \mathcal{B}_x$, then there is some $V_3 \in \mathcal{B}_x$ such that $V_3 \subset V_1 \cap V_2$,

V-c) if $V \in \mathcal{B}_x$, there is some $V_0 \in \mathcal{B}_x$ such that, for any $y \in V_0$, there is some $W \in \mathcal{B}_y$ with $W \subset V$,

and if we define a set G to be "open" if and only if it contains a basic neighborhood of each of its points, then the result is a topology on X in which $\mathcal{B}_x$ is a neighborhood base at x, for each $x \in X$.

Let $\mathcal{B}_x$ denote the neighborhood base of a point $x \in \Gamma$ given by the Moore plane. We proceed by verifying that $\mathcal{B}_x$ satisfies each of the above properties for each $x \in \Gamma$.

Claim 36. If $V \in \mathcal{B}_x$, then $x \in V$.

Proof. Let $V \in \mathcal{B}_x$. Either V is an open ball centered at x, or V is an open ball tangent to x together with x itself. In either case, we see that $x \in V$, as desired. $\square$

Claim 37. If $V_1, V_2 \in \mathcal{B}_x$, then there is some $V_3 \in b_x$ such that $V_3 \subset V_1 \cap V_2$.

Proof. Let V_1 and V_2 belong to $\mathcal{B}_x$. We consider two cases.

Case x lies on the real line

As x lies on the real line, it must be that V_1 and V_2 are both tangent to x. Furthermore, we have that one is contained in the other. Without loss of generality, let $V_1 \subset V_2$. Choosing V_3 to be V_1, we have $V_3 = V_1 \in \mathcal{B}_x$ and $V_3 = V_1 \subset V_1 \cap V_2$, as desired.

Case x lies strictly in the upper half-plane

As x lies strictly in the upper half-plane, it must be that V_1 and V_2 are both open balls centered at x. Furthermore, we have that one is contained in the other. Without loss of generality, let $V_1 \subset V_2$. Choosing V_3 to be V_1, we have $V_3 = V_1 \in \mathcal{B}_x$ and $V_3 = V_1 \subset V_1 \cap V_2$, as desired. $\square$

Claim 38. If $V \in \mathcal{B}_x$, there is some $V_0 \in b_x$ such that, for any $y \in V_0$, there is some $W \in b_y$ with $W \subset V$.

Proof. Let $V \in \mathcal{B}_x$. We consider two cases.

Case x lies on the real line

As x lies on the real line, it must be that V is tangent to x. Choose V_0 to be V. If y is chosen to be x, then taking $W = V$ suffices. Otherwise, y lies strictly inside the open ball $V - \{x\}$, and so there will always be a smaller open ball centered at y and contained in V (which we will take to be W).

Case x lies strictly in the upper half-plane

As x lies strictly in the upper half-plane, it must be that V is an open ball centered at x. Choose V_0 to be V. As V is open, for any $y \in V_0 = V$, there is a smaller open ball centered at y and contained in V (which we will take to be W). □

Therefore, the Moore plane gives a topology on Γ. □

12 Extra Problem 1

Proposition 12.1. *Let (X, τ) and (Y, σ) be topological spaces, and let $\mathcal{B}$ be a base for τ. The function $f : Y \to X$ is continuous if and only if $f^{\leftarrow}B \in \sigma$ for all $B \in \mathcal{B}$.*

Proof. ($\Rightarrow$) Suppose f is continuous. It follows that

$$\begin{aligned} f \text{ is continuous} \Rightarrow \quad & f^{\leftarrow}U \text{ is open for every open set } U \subset X \\ \Rightarrow \quad & f^{\leftarrow}B \text{ is open for every } B \in \mathcal{B} \qquad \text{(as each } B \in \mathcal{B} \text{ is itself an open set)} \\ \Rightarrow \quad & f^{\leftarrow}B \in \sigma \text{ for every } B \in \mathcal{B} \end{aligned}$$

($\Leftarrow$) Suppose $f^{\leftarrow}B \in \sigma$ for all $B \in \mathcal{B}$. Let U be any open subset of X. Our goal is to show that $f^{\leftarrow}U$ is open in Y, thus establishing the continuity of f.

Since $\mathcal{B}$ is a base for τ, there exists a collection $\mathcal{C} \subset \mathcal{B}$ such that $U = \bigcup\{B \in \mathcal{C}\}$. Now,

$$\begin{aligned} f^{\leftarrow}U &= f^{\leftarrow}\left(\bigcup\{B \in \mathcal{C}\}\right) \\ &= \bigcup_{B \in \mathcal{C}} f^{\leftarrow}B. \end{aligned}$$

By hypothesis, each $f^{\leftarrow}B$ belongs to σ. In other words, each $f^{\leftarrow}B$ is open in Y, and so the union of these sets is open in Y, as well. Thus, we have shown that $f^{\leftarrow}U$ is open in Y for any open $U \subset X$. Therefore, f is continuous. □

13 Problem 6A1

Proposition 13.1. *Let $\mathcal{B}A$ denote the slotted plane. Any straight line in the plane has the discrete topology as a subspace of $\mathcal{B}A$.*

Proof. Let τ be the relative topology on any line $L \subset \mathbb{R}^2$ as a subspace of $\mathcal{B}A$. Observe first that we can find basic neighborhoods in $\mathcal{B}A$ which intersect trivially with L, and so we can construct $\emptyset$. Now, for any point $x \in L$, consider any basic neighborhood $\{x\} \cup A$ in $\mathcal{B}A$ where we require that one of the lines removed from the open disk A coincides with L. Under this constraint, $(\{x\} \cup A) \cap L = \{x\}$. Since we can construct any isolated point by intersecting L with some basic neighborhood in $\mathcal{B}A$, we can take unions to construct any subset of L. Therefore, any subset of L is open in τ, and so τ is the discrete topology. □

Proposition 13.2. *Let $\mathcal{B}A$ denote the slotted plane. The topology on any circle in the plane as a subspace of $\mathcal{B}A$ coincides with its usual topology.*

Proof. Let C be any circle in $\mathbb{R}^2$. Denote the usual topology of C by σ and the relative topology as a subspace of $\mathcal{B}A$ by τ. We show that $\sigma = \tau$.

($\subset$) Let O be a basic open set in σ. That is, O an open interval lying on C. Furthermore, $O = G \cap C$, where G is an open Euclidean ball in $\mathbb{R}^2$. As G is an open ball with finitely-many (i.e. zero) lines removed, $G \in \mathcal{B}A$. Therefore, $O = G \cap C \in \tau$.

($\supset$) Let O be a basic open set in τ. That is, $O = C \cap A$ for some $A \in \mathcal{B}A$. If $A = \emptyset$, then $C \cap \emptyset = \emptyset \in \sigma$. If $A = \mathbb{R}^2$, then $C \cap \mathbb{R}^2 = C \in \sigma$. Otherwise, $C \cap A$ is the finite union of disjoint open intervals lying on C (the open disk of A selects some open interval of C, while the finite number of removed lines subdivides this into a finite number of open subintervals). In this case, we still have $C \cap A \in \sigma$, as desired. $\square$

14 Problem 6A2

Proposition 14.1. *Let $\mathcal{B}B$ denote the radial plane. The relative topology induced on any straight line as a subspace of $\mathcal{B}B$ is its usual topology.*

Proof. Let L be any line in the plane. Denote the topology of L inherited from the usual topology on the plane by σ and the relative topology of L as a subspace of $\mathcal{B}B$ by τ. We show that $\sigma = \tau$.

($\subset$) Let O be a basic open set of σ. That is,

$$\begin{aligned} O &= (x - \epsilon, x + \epsilon) \\ &= L \cap (x - \epsilon, x + \epsilon) \\ &= L \cap B(x, \epsilon). \end{aligned}$$

Hence, O is an element of τ.

($\supset$) Let O be a basic open set of τ. That is, O is the union of a collection of open line segments centered around some point $x \in \mathbb{R}^2$. If O intersects trivially with L, then $L \cap O = \emptyset \in \sigma$. If x lies on L, O contains an open line segment centered at x coinciding with an open interval of L. Hence, $L \cap O$ is an open interval lying on L, and so $L \cap O \in \sigma$. $\square$

Proposition 14.2. *Let $\mathcal{B}B$ denote the radial plane. The relative topology on any circle in the plane as a subspace of $\mathcal{B}B$ is the discrete topology.*

Proof. Let C be any circle in the plane. Denote the relative topology of C as a subspace of $\mathcal{B}B$ by τ. For $x \notin C$, we can always find an open ball about x of small enough radius that intersects trivially with C. Hence, $\emptyset \in \tau$. Now, for any $x \in C$, we wish to find an open set $O \in \mathcal{B}B$ such that $C \cap O = \{x\}$. We require that O possess an open line segment about x in each direction. We claim that $O = B(x, \epsilon) \setminus C$ suffices, where ϵ is the radius of C. To see this, consider the line passing through x in a given direction. If the line is tangent to x, we have an open neighborhood of x of length 2ϵ. Otherwise, the line is a chord of C, and so intersects C at some point y. In this direction, we have any open neighborhood of length $\epsilon + d(x, y)$. Hence, O is radially open about x and is constructed in such a way that $C \cap O = \{x\}$. By taking unions, we see that any subset of C is open in the relative topology τ, and so τ is the discrete topology. $\square$

15 Problem 6C

Proposition 15.1. *If M is metrizable and $N \subset M$, then the subspace N is metrizable with the topology generated by the restriction of any metric which generates the topology on M.*

Proof. Let τ be the topology on M generated by a metric ρ. Let σ be the relative topology on N and let ρ_N be the restriction of ρ to N. We show that σ is generated by ρ_N.

Let $O \in \sigma$. It must be that $O = N \cap G$ for some $G \in \tau$. Since M is generated by ρ, we know that $G = \bigcup_{x \in G} B_\rho(x, \epsilon_x)$, where $\epsilon_x > 0$ may depend on x. Now,

$$\begin{aligned} O &= N \cap G \\ &= N \cap \bigcup_{x \in G} B_\rho(x, \epsilon_x) \\ &= \bigcup_{x \in G} N \cap B_\rho(x, \epsilon_x) \\ &= \bigcup_{x \in N \cap G} B_{\rho_N}(x, \epsilon_x). \end{aligned}$$

Hence, O is the union of open balls with respect to the metric ρ_N. Therefore, σ is generated by the ρ_N, as desired. □

16 Extra Problem

Proposition 16.1. *Let X and Y be topological spaces and $f : X \to Y$ a bijection. The following are equivalent:*

a.) The function f is a homeomorphism.

b.) For any $G \subset X$, $f^{\to}G$ is open in Y if and only if G is open in X.

c.) For any $F \subset X$, $f^{\to}F$ is closed in Y if and only if F is closed in X.

d.) For any $E \subset X$, $f^{\to}\overline{E} = \overline{f^{\to}E}$.

Proof. (We have shown $a \Leftrightarrow b \Leftrightarrow c$ in class.)

((a and b) $\Rightarrow d$) Let f be a homeomorphism (and so also possesses the property that, for any $G \subset X$, $f^{\to}G$ is open in Y if and only if G is open in X).

We show first that $f^{\to}\overline{E} \subset \overline{f^{\to}E}$. To that end, let $b \in f^{\to}\overline{E}$ and consider any open O containing b. By the continuity of f, $f^{\leftarrow}O$ is open. Furthermore, there is an element a of $f^{\leftarrow}O$ such that $f(a) = b$. Now, since $a \in \overline{E}$, any open set containing a intersects nontrivially with E. In particular, the open set $f^{\leftarrow}O$ intersects nontrivially with E, and so $f^{\to}(f^{\leftarrow}O)$ intersects nontrivially with $f^{\to}E$. As f is a bijection, $f^{\to}(f^{\leftarrow}O) = O$, and so we see that O intersects nontrivially with $f^{\to}E$. In other words, $b \in \overline{f^{\to}E}$, as desired.

Next we show that $\overline{f^{\to}E} \subset f^{\to}\overline{E}$. To that end, let $b \in \overline{f^{\to}E}$ and suppose, for the purpose of contradiction, that $b \notin f^{\to}\overline{E}$. In other words, $b = f(a)$ for $a \notin \overline{E}$ (such an a must exist, since f is a bijection). Hence, we can find an open ball O containing a such that O intersects trivially with E, which implies that $f^{\to}O$ is an open set containing b that intersects trivially with $f^{\to}E$. In other words, $b \notin \overline{f^{\to}E}$, which is a contradiction. Therefore, $\overline{f^{\to}E} \subset f^{\to}\overline{E}$.

($d \Rightarrow c$) Let f be such that, for any $E \subset X$, $f^{\to}\overline{E} = \overline{f^{\to}E}$.

We show first that, for any $F \subset X$, $f^{\to}F$ is closed in Y implies that F is closed in X. For that, observe

$$\begin{aligned} f^{\to}F \text{ is closed in } Y &\Rightarrow f^{\to}F = \overline{f^{\to}F} \\ &\Rightarrow f^{\to}F = f^{\to}\overline{F} \\ &\Rightarrow F = \overline{F} && \text{(since } f \text{ is a bijection)} \\ &\Rightarrow F \text{ is closed in } X. \end{aligned}$$

We show next that, for any $F \subset X$, F is closed in X implies that $f^{\to}F$ is closed in Y. For that, observe

$$\begin{aligned} F \text{ is closed in } X &\Rightarrow F = \overline{F} \\ &\Rightarrow f^{\to}F = f^{\to}\overline{F} \\ &\Rightarrow f^{\to}F = \overline{f^{\to}F} \\ &\Rightarrow f^{\to}F \text{ is closed in } Y. \end{aligned}$$

Therefore, for any $F \subset X$, $f^{\to}F$ is closed in Y if and only if F is closed in X. □

17 Problem 7A

Definition 17.1. The <u>characteristic function</u> of a subset A of a set X (denoted χ_A) is the function from X to $\mathbb{R}$ which is 1 at points of A and 0 at other points of X.

Proposition 17.2. *The characteristic function of A is continuous if and only if A is both open and closed in X.*

Proof. (⇒) Let χ_A be continuous. Observe $\chi_A : X \to \{0,1\}$ is indeed a function between topological spaces when each subset of $\mathbb{R}$ is taken together with its relative topology. Now,

$$f^{\leftarrow}(\{1\}) = A.$$

As $\{1\}$ is closed in $\{0,1\}$, A is closed in X (by the continuity of χ_A). Similarly, we have

$$f^{\leftarrow}(\{0\}) = A^c.$$

As $\{0\}$ is closed in $\{0,1\}$, A^c is closed in X (by the continuity of χ_A), and so A is open in X.

(⇐) Let A be both open and closed in X and consider an arbitrary open subset O of $\mathbb{R}$. We show that $f^{\leftarrow}(B \cap O)$ is open for all subsets B of $\{0,1\}$.

$$\begin{aligned} f^{\leftarrow}(\emptyset) &= \emptyset, \text{ which is open} \\ f^{\leftarrow}(\{0\}) &= A^c, \text{ which is open, since } A \text{ is closed} \\ f^{\leftarrow}(\{1\}) &= A, \text{ which is open} \\ f^{\leftarrow}(\{0,1\}) &= X, \text{ which is open} \end{aligned}$$

Therefore, χ_A is continuous. □

Proposition 17.3. *The topological space X has the discrete topology if and only if $f : X \to Y$ is continuous whenever (Y, τ) is a topological space.*

Proof. (⇒) Let X have the discrete topology. That is, every subset A of X is open. In particular, $f^{\leftarrow}(O)$ is open in X for any open subset O of Y. Therefore, f is continuous.

(⇐) Let $f : X \to Y$ be continuous whenever (Y, τ) is a topological space. In particular, let $Y = X$, τ be the discrete topology, and f be the identity on X. Now, for any subset A of the codomain, we have that A is open (since the codomain has the discrete topology). Hence, by the continuity of f, $f^{\leftarrow}(A)$ is open. At the same time, by the definition of f, $f^{\leftarrow}(A) = A$. Therefore, every subset A of the domain is open, and so X has the discrete topology. □

Proposition 17.4. *The topological space X has the trivial topology if and only if $f : Y \to X$ is continuous whenever (Y, τ) is a topological space.*

Proof. (⇒) Let X have the trivial topology. To establish the continuity of f, we show that the preimage of any open set in X is open in Y. As X has the trivial topology, it suffices to observe that

$$\begin{aligned} f^{\leftarrow}(\emptyset) &= \emptyset \text{ which is open in } Y \\ f^{\leftarrow}(X) &= Y \text{ which is open in } Y. \end{aligned}$$

Therefore, f is continuous.

(⇐) Let $f : Y \to X$ be continuous whenever (Y, τ) is a topological space. In particular, let $Y = X$, τ be the trivial topology, and f be the identity on X. Now, consider any open set A in X. Since f is continuous, $f^{\leftarrow}(A)$ is open in Y. As the domain has the trivial topology, it must be that $f^{\leftarrow}(A)$ is either the empty set or all of X. At the same time, since f is the identity on X, we see that A is either the empty set or all of X. Therefore, the only open sets in the codomain are the empty set or X, and so the codomain has the trivial topology. □

18 Problem 6B3

Proposition 18.1. *An open subset of a separable space is separable.*

Proof. Let O be an open subset of the separable space X, and let D be the countable, dense set in X. Consider any open neighborhood G of a point $x \in O$ with the constraint that G be contained in O (since O is open, this can always be done). Since D is dense in X, $G \cap D \neq \emptyset$. Hence, any open neighborhood of a point in O contains an element of D. In other words, D is dense in O. □

19 Problem 7B

Recall that the Cantor-Bernstein theorem states that if A and B are sets and if one-to-one functions $f : A \to B$ and $g : B \to A$ exist, then a one-to-one function of A onto B exists. The analog for topological spaces would be as follows: Whenever X can be embedded in Y and Y can be embedded in X, then X and Y are homeomorphic.

Proposition 19.1. *The aforementioned analog of the Cantor-Bernstein theorem for topological spaces is false.*

Proof. Define $\mathbb{R}^{\geq 1}$ to be the set $\{x \in \mathbb{R} \mid x \geq 1\}$.

Claim 39. The function

$$f : [0,1] \to \mathbb{R}^{\geq 1}$$
$$f(x) = x + 1$$

is an embedding of $[0,1]$ into $\mathbb{R}^{\geq 1}$.

Proof. The function f is one-to-one and continuous. The inverse of f, given by $f^{-1}(x) = x - 1$, is also continuous. □

Claim 40. The function

$$g : \mathbb{R}^{\geq 1} \to [0,1]$$
$$g(x) = \frac{1}{x}$$

is an embedding of $\mathbb{R}^{\geq 1}$ into $[0,1]$.

Proof. The function g is one-to-one and continuous. The inverse of g, given by $g^{-1}(x) = x$, is also continuous. □

Now, the property that every continuous, real-valued function on some set achieves its maximum is a topology property. As $[0,1]$ is a compact set, it possesses this property. The property does not hold, however, for $\mathbb{R}^{\geq 1}$ (the identity function on $\mathbb{R}^{\geq 1}$ serves as a counterexample). Therefore, while there exists an embedding of $[0,1]$ into $\mathbb{R}^{\geq 1}$ and vice versa, the two spaces are not homeomorphic, and so the proposed analog of the Cantor-Bernstein theorem is not true in general. □

20 Problem 8D

Let X and Y be topological spaces containing subsets A and B, respectively.

Proposition 20.1. *In the product space $X \times Y$, $(A \times B)^\circ = A^\circ \times B^\circ$.*

Proof. It follows directly that

$$\begin{aligned}
(x,y) \in (A \times B)^\circ &\Leftrightarrow (x,y) \in G \text{ for some open set } G \subset A \times B \\
&\Leftrightarrow (x,y) \in G_1 \times G_2 \text{ for some open sets } G_1 \subset A \text{ and } G_2 \subset B \\
&\Leftrightarrow x \in G_1 \text{ and } y \in G_2 \text{ for some open sets } G_1 \subset A \text{ and } G_2 \subset B \\
&\Leftrightarrow x \in A^\circ \text{ and } y \in B^\circ \\
&\Leftrightarrow (x,y) \in A^\circ \times B^\circ.
\end{aligned}$$

□

Proposition 20.2. *In the product space* $X \times Y$, $\overline{A \times B} = \overline{A} \times \overline{B}$.

Proof. To see that $\overline{A \times B} \subset \overline{A} \times \overline{B}$, observe that

$$\begin{aligned}
(x, y) \in \overline{A \times B} &\Rightarrow (x, y) \in F \text{ for all closed sets } F \supset A \times B \\
&\Rightarrow (x, y) \in F_1 \times F_2 \text{ for all closed sets } F_1 \supset A \text{ and } F_2 \supset B \\
&\Rightarrow x \in F_1 \text{ and } y \in F_2 \text{ for all closed sets } F_1 \supset A \text{ and } F_2 \supset B \\
&\Rightarrow x \in \overline{A} \text{ and } y \in \overline{B} \\
&\Rightarrow (x, y) \in \overline{A} \times \overline{B}.
\end{aligned}$$

To see that $\overline{A} \times \overline{B} \subset \overline{A \times B}$, observe that

$$\begin{aligned}
(x, y) \in \overline{A} \times \overline{B} &\Rightarrow x \in \overline{A} \text{ and } y \in \overline{B} \\
&\Rightarrow \text{ for all basic open } O_x \text{ containing } x,\ O_x \cap A \neq \emptyset, \\
&\qquad \text{and for all basic open } O_y \text{ containing } y,\ O_y \cap B \neq \emptyset \\
&\Rightarrow \text{ for all basic open } O \text{ containing } (x, y),\ O \cap (A \times B) \neq \emptyset \\
&\Rightarrow (x, y) \in \overline{A \times B}.
\end{aligned}$$

□

21 Problem 8D3

Proposition 21.1. *Let* X_α *be topological spaces containing subsets* A_α*, respectively, for* $\alpha \in \Gamma$*. In the product space* $\prod_{\alpha \in \Gamma} X_\alpha$*, we have*

$$\overline{\prod_{\alpha \in \Gamma} A_\alpha} = \prod_{\alpha \in \Gamma} \overline{A_\alpha}.$$

Proof. Recall that, for any set subset E of a topological space X,

$$\overline{E} = \{x \in X \mid \text{each basic neighborhood of } x \text{ meets } E\}.$$

It follows that,

$$\begin{aligned}
\overline{\prod_{\alpha \in \Gamma} A_\alpha} &= \left\{ x \in \prod_{\alpha \in \Gamma} X_\alpha \mid \text{each basic neighborhood of } x \text{ meets } \prod_{\alpha \in \Gamma} A_\alpha \right\} \\
&= \left\{ x \in \prod_{\alpha \in \Gamma} X_\alpha \mid \text{for all } \alpha \in \Gamma, \text{ each basic neighborhood of } x_\alpha \text{ meets } A_\alpha \right\} \\
&= \left\{ x \in \prod_{\alpha \in \Gamma} X_\alpha \mid \text{for all } \alpha \in \Gamma, \text{ each basic neighborhood of } x_\alpha \in \overline{A_\alpha} \right\} \\
&= \prod_{\alpha \in \Gamma} \overline{A_\alpha}.
\end{aligned}$$

□

Remark 21.2. The proposition above is not true in general for the interior operation. For a counterexample, let $A_\alpha = (0, 1)$ for $\alpha \in \Gamma$ and consider them as subspaces of $\mathbb{R}^\omega$ with the usual topology. We have

$$\begin{aligned}
\left(\prod_{\alpha \in \Gamma} (0, 1) \right)^\circ &= \bigcup \left\{ G \mid G \text{ is open, } G \subset \prod_{\alpha \in \Gamma} (0, 1) \right\} \\
&= \emptyset.
\end{aligned}$$

To see why this is the case, recall that the open sets in this product topology must equal $\mathbb{R}$ at all but finitely-many coordinates. Hence, the only open set contained in $\prod_{\alpha\in\Gamma} A_\alpha$ is the empty set.

On the other hand, we have

$$\begin{aligned}\prod_{\alpha\in\Gamma}(0,1)^\circ &= \prod_{\alpha\in\Gamma}(0,1)\\ &\neq \emptyset.\end{aligned}$$

22 Problem 8H1

Proposition 22.1. *Let X have the weak topology induced by a collection of maps $f_\alpha : X \to X_\alpha$ for $\alpha \in \Gamma$. If each X_α has the weak topology given by a collection of maps $g_{\alpha\lambda} : X_\alpha \to Y_{\alpha\lambda}$, for $\lambda \in \Lambda_\alpha$, then X has the weak topology given by the maps $g_{\alpha\lambda} \circ f_\alpha : X \to Y_{\alpha\lambda}$, for $\alpha \in \Gamma$ and $\lambda \in \Lambda_\alpha$.*

Proof. We first verify that the open sets in X are indeed sufficient to make the functions $g_{\alpha\lambda} \circ f_\alpha$ continuous for all $\alpha \in \Gamma$ and $\lambda \in \Lambda$. To that end, choose an open set $O_{\alpha\lambda} \in Y_{\alpha\lambda}$. It follows that

$$(g_{\alpha\lambda} \circ f_\alpha)^{-1}(O_{\alpha\lambda}) = f_\alpha^{-1}(g_{\alpha\lambda}^{-1}(O_{\alpha\lambda})).$$

Since $g_{\alpha\lambda}$ is continuous, $g_{\alpha\lambda}^{-1}(O_{\alpha\lambda})$ is open. Since f_α is continous, the preimage of this open set is itself open. Hence, $g_{\alpha\lambda} \circ f_\alpha$ is continous.

Evidently, the open sets in X are also necessary to make the functions $g_{\alpha\lambda} \circ f_\alpha$ continuous for all $\alpha \in \Gamma$ and $\lambda \in \Lambda$ (i.e. they form the weak topology on X induced by $g_{\alpha\lambda} \circ f_\alpha$). To that end, let $O \in X$. Since the f_α induce the weak topology on X, there exists some O_α such that $f^{-1}(O_\alpha) = O$. Furthermore, since the g_{α_λ} induce the weak topology on X_α, there exists some $O_{\alpha\lambda}$ such that $g_{\alpha\lambda}^{-1}(O_{\alpha\lambda}) = O_\alpha$. Hence,

$$\begin{aligned}(g_{\alpha\lambda} \circ f_\alpha)^{-1}(O_{\alpha\lambda}) &= f_\alpha^{-1}(g_{\alpha\lambda}^{-1}(O_{\alpha\lambda}))\\ &= f_\alpha^{-1}(O_\alpha)\\ &= O.\end{aligned}$$

As the open sets in X are both necessary and sufficient to make the functions $g_{\alpha\lambda} \circ f_\alpha$ continuous for all $\alpha \in \Gamma$ and $\lambda \in \Lambda$, we conclude that X indeed has the weak topology induced by $g_{\alpha\lambda} \circ f_\alpha$. ☐

23 Problem 9B1

Definition 23.1. Let X be a topological space. A decomposition $\mathcal{D}$ of X is a collection of disjoint subsets of X whose union is X. If a decomposition $\mathcal{D}$ is endowed with the topology in which $\mathcal{F} \subset \mathcal{D}$ is open if and only if

$$\bigcup\{F \mid F \in \mathcal{F}\}$$

is open in X, then $\mathcal{D}$ is referred to as a decomposition space of X.

Proposition 23.2. *The process given above for forming the topology on a decomposition space does indeed define a topology.*

Proof. Let τ be the proposed topology on $\mathcal{D}$. We verify that τ satisfies each of the properties of a topology.

Claim 41. If $\mathcal{F}_\alpha$ belongs to τ for all $\alpha \in \Gamma$, then so does $\bigcup_{\alpha\in\Gamma} \mathcal{F}_\alpha$.

Proof. By definition, each $\mathcal{F}_\alpha$ is itself a union of sets that are open in X. Denote of consituent open sets of $\mathcal{F}_\alpha$ by $F_{\alpha\beta}$ with $\beta \in \Gamma'$. It follows that

$$\begin{aligned}\bigcup_{\alpha\in\Gamma} \mathcal{F}_\alpha &= \bigcup_{\alpha\in\Gamma}\bigcup_{\beta\in\Gamma'} F_{\alpha\beta}\\ &= \bigcup_{\alpha\in\Gamma,\ \beta\in\Gamma'} F_{\alpha\beta}.\end{aligned}$$

Since each of the $F_{\alpha\beta}$ is open in X, their union is also open in X. Therefore, $\bigcup_{\alpha\in\Gamma} \mathcal{F}_\alpha$ belongs to τ. ☐

Claim 42. If $\mathcal{F}_i$ belongs to τ for all $1 \leq i \leq n$, then so does $\bigcap_{i=1}^n \mathcal{F}_i$.

Proof. By definition, each $\mathcal{F}_i$ is itself a union of sets that are open in X. Denote of consituent open sets of $\mathcal{F}_i$ by $F_{i\alpha}$ with $\alpha \in \Gamma$. It follows that

$$\bigcap_{i=1}^n \mathcal{F}_i = \bigcap_{i=1}^n \bigcup_{\alpha\in\Gamma} F_{i\alpha}.$$

Since each of the $F_{i\alpha}$ is open in X, their union is also open in X. Since the finite intersetion of open sets is again open, we have that $\bigcap_{i=1}^n \mathcal{F}_i$ belongs to τ, as desired. □

Claim 43. The sets $\emptyset$ and $\mathcal{D}$ belong to τ.

Proof. We have that

$$\begin{aligned}\bigcup\{F : F \in \emptyset\} &= \bigcup \emptyset \\ &= \emptyset,\end{aligned}$$

which is open in X. Hence, $\emptyset$ belongs to τ.

We also have that

$$\bigcup\{F : F \in \mathcal{D}\} = X,$$

as $\mathcal{D}$ is a decomposition of X. Since X is open in X, we have that $\mathcal{D}$ belongs to τ. □

Therefore, τ is a topology on $\mathcal{D}$. □

24 Problem 9B2

Let $\mathcal{D}$ denote the decomposition space of a topological space X and let P be the natural map from X onto $\mathcal{D}$.

Lemma 24.1. *For any subset $\mathcal{F}$ of $\mathcal{D}$, $P^{\leftarrow}(\mathcal{F}) = \bigcup\{F \mid F \in \mathcal{F}\}$.*

Proof.

$$\begin{aligned}x \in P^{\leftarrow}(\mathcal{F}) &\Leftrightarrow P^{\rightarrow}(x) = F, \text{ some } F \in \mathcal{F} \\ &\Leftrightarrow x \in F, \text{ some } F \in \mathcal{F} \\ &\Leftrightarrow x \in \bigcup\{F \mid F \in \mathcal{F}\}\end{aligned}$$

□

Proposition 24.2. *The topology on $\mathcal{D}$ is the quotient topology induced by P.*

Proof. Let τ denote the topology on the decomposition space $\mathcal{D}$ and let τ_P denote the quotient topology induced by the natural map. It follows that

$$\begin{aligned}\mathcal{O} \in \tau &\Leftrightarrow \mathcal{O} = \{F_\alpha \mid F_\alpha \subset X,\ \alpha \in \Gamma\} \text{ with } \bigcup_{\alpha\in\Gamma} F_\alpha \text{ open in } X \\ &\Leftrightarrow P^{\leftarrow}(\mathcal{O}) \text{ open in } X \qquad \left(\text{as } P^{\leftarrow}(\mathcal{O}) = \bigcup_{\alpha\in\Gamma} F_\alpha \text{ by 24.1}\right) \\ &\Leftrightarrow \mathcal{O} \in \tau_P.\end{aligned}$$

Therefore, $\tau = \tau_P$. □

25 Problem 9C2

Proposition 25.1. *A closed, continuous, onto map need not be open.*

Proof. Consider the function f mapping the interval $[-\pi, 3\pi]$ to the unit circle C by $f(x) = (\cos x, \sin y)$. It is easily seen that f is closed (the image of a closed subinterval of $[-\pi, 3\pi]$ is a closed arc of C), continuous (the preimage of an open arc in C is an open subinterval of $[-\pi, 3\pi]$), and onto ($f^{\rightarrow}[0, 2\pi] = C$).

(I cannot work out how to make use of this counterexample suggested by Willard. It seems that, for any open subinterval O, $f^{\rightarrow}(O)$ is an open arc of C. Hence, a general open set in $[-\pi, 3\pi]$ will simply be the union of open sets in C, which is again open. By the way, I chose $[-\pi, 3\pi]$ as the domain, since any bijection is necessarily both open and closed. Hence, I was hoping to exploit the fact that my choice of domain causes f to fail to be injective to create my counterexample.) □

26 Problem 9H1

Definition 26.1. Suppose X_α is a topological space and f_α is a map of X_α to a set Y for each $\alpha \in A$. The strong topology coinduced by the maps f_α on Y consists of all sets U in Y such that $f_\alpha^{-1}(U)$ is open in X_α for each $\alpha \in A$.

Proposition 26.2. *The strong topology is indeed a topology. Moreover, it is the largest topology making each f_α continuous.*

Proof. Let τ be the proposed topology on Y. We verify that τ satisfies each of the properties of a topology.

Claim 44. If O_β belongs to τ for all $\beta \in \Gamma$, then so does $\bigcup_{\beta\in\Gamma} O_\beta$.

Proof. It follows immediately that

$$\begin{aligned} O_\beta \in \tau \text{ for all } \beta \in \Gamma &\Rightarrow f_\alpha^{-1}(O_\beta) \text{ is open in } X_\alpha \text{ for all } \alpha \in A \text{ and } \beta \in \Gamma \\ &\Rightarrow \bigcup_{\beta\in\Gamma} f_\alpha^{-1}(O_\beta) \text{ is open in } X_\alpha \text{ for all } \alpha \in A \\ &\Rightarrow f_\alpha^{-1}\left(\bigcup_{\beta\in\Gamma} O_\beta\right) \text{ is open in } X_\alpha \text{ for all } \alpha \in A \qquad \left(\text{as } f_\alpha^{-1}\left(\bigcup_{\beta\in\Gamma} O_\beta\right) = \bigcup_{\beta\in\Gamma} f_\alpha^{-1}(O_\beta)\right) \\ &\Rightarrow \bigcup_{\beta\in\Gamma} O_\beta \in \tau. \end{aligned}$$

□

Claim 45. If O_i belongs to τ for all $1 \le i \le n$, then so does $\bigcap_{i=1}^n O_i$.

Proof. (I am stumped here. The best I can claim is that, for all $\alpha \in A$,

$$f_\alpha^{-1}\left(\bigcap_{i=1}^n O_i\right) \subset \bigcap_{i=1}^n f_\alpha^{-1}(O_i)$$

We have that the righthand side is open in X_α, since each element of the finite intersection is open by hypothesis. I do not see that this forces the lefthand side to be open, as well.) □

Claim 46. The sets $\emptyset$ and Y belong to τ.

Proof. For all $\alpha \in A$,

$$f_\alpha^{-1}(\emptyset) = \emptyset,$$

which is open in X_α. Hence, $\emptyset \in \tau$.

For all $\alpha \in A$,

$$f_\alpha^{-1}(Y) = X_\alpha,$$

which is open in X_α. Hence, $Y \in \tau$. □

Therefore, τ is indeed a topology on Y. To see that this is the largest topology making each f_α continuous, consider any other topology σ making each f_α continuous. Given any open set O belonging to σ, we have that $f_\alpha^{-1}(O)$ is open for all $\alpha \in A$ (as each f_α is continuous). By definition of openness in τ, we have that $O \in \tau$. Hence, $\sigma \subset \tau$, and so τ is indeed the largest topology making each f_α continuous. □

27 Problem 13B1

Proposition 27.1. *Any subspace of a T_0-space is T_0.*

Proof. Let X be a T_0-space and let A be a subspace of X. For any distinct $x, y \in A \subset X$, there is a set G that is open in X such that

$$x \in G \text{ and } y \notin G, \text{ or}$$
$$x \notin G \text{ and } y \in G,$$

since X is a T_0-space. Now, consider the set $G \cap A$, which is open in A. It follows that

$$x \in G \text{ and } y \notin G, \text{ or}$$
$$x \notin G \text{ and } y \in G,$$

Therefore, A is a T_0-space. □

Proposition 27.2. *Any subspace of a T_1-space is T_1.*

Proof. Let X be a T_1-space and let A be a subspace of X. For any distinct $x, y \in A \subset X$, there is a set G that is open in X such that

$$x \in G \text{ and } y \notin G,$$

since X is a T_1-space. Now, consider the set $G \cap A$, which is open in A. It follows that

$$x \in G \cap A \text{ and } y \notin G \cap A.$$

Therefore, A is a T_1-space. □

28 Problem 13B2

Proposition 28.1. *Any nonempty product space is T_1 if and only if each factor space is T_1.*

Proof. ($\Rightarrow$) Let $\prod_{\alpha\in\Gamma} X_\alpha$ be T_1 and let x' and y' be distinct points in some particular X_β. Consider elements $x, y \in \prod_{\alpha\in\Gamma} X_\alpha$ which are equal to x' and y' (respectively) in the β^{th} coordinate and have $x_\alpha = y_\alpha$ for all $\alpha \neq \beta$. As $\prod_{\alpha\in\Gamma} X_\alpha$ is T_1, it follows that there is an open set $\prod_{\alpha\in\Gamma} G_\alpha$ such that $x \in \prod_{\alpha\in\Gamma} G_\alpha$ and $y \notin \prod_{\alpha\in\Gamma} G_\alpha$. Hence, $x_\alpha \in G_\alpha$ for all $\alpha \in \Gamma$, so, in particular, $x' \in G_\beta$. Now, it must be that $y' \notin G_\beta$, as $y_\alpha = x_\alpha \in G_\alpha$ for all $\alpha \neq \beta$. Hence, the set G_β is an open set in X_β with $x' \in G_\beta$ and $y' \notin G_\beta$. That is, X_β is T_1. As β was chosen arbitrarily, it follows that each factor space is T_1.

($\Leftarrow$) Let X_α be T_1 for all $\alpha \in \Gamma$ and let x, y be distinct points in $\prod_{\alpha\in\Gamma} X_\alpha$. Since each factor space is T_1, we can find, for all $\alpha \in \Gamma$, $G_\alpha \subset X_\alpha$ such that $x_\alpha \in G_\alpha$ and $y_\alpha \notin G_\alpha$. It follows that $x \in \prod_{\alpha\in\Gamma} G_\alpha$ and $y \notin \prod_{\alpha\in\Gamma} G_\alpha$. Therefore, $\prod_{\alpha\in\Gamma} X_\alpha$ is T_1. □

29 Problem 13D1

For a polynomial P in n real variables, let $Z(P) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid P(x_1, \ldots, x_n) = 0\}$. Let $\mathcal{P}$ be the collection of all such polynomials.

Proposition 29.1. *$\{Z(P) \mid P \in \mathcal{P}\}$ is a base for the closed sets of a topology (the Zariski topology) on $\mathbb{R}^n$.*

Proof. We show that $\bigcap_{P\in\mathcal{P}} Z(P) = \emptyset$ and that, for any $P_1, P_2 \in \mathcal{P}$, $Z(P_1) \cup Z(P_2)$ is equal to the intersection of some subfamily of $\{Z(P) \mid P \in \mathcal{P}\}$, thereby establishing that $\{Z(P) \mid P \in \mathcal{P}\}$ is a base for the closed sets of a topology on $\mathbb{R}^n$.

Let $P_0 \in \mathcal{P}$ denote the polynomial in n variables whose output is 1 for any input. As P_0 has no roots, it must be that $Z(P_0) = \emptyset$. Hence, $\bigcap_{P\in\mathcal{P}} Z(P) = \emptyset$.

Let $P_1, P_2 \in \mathcal{P}$. We claim that $Z(P_1) \cup Z(P_2) = Z(P_1P_2)$, which belongs to $\{Z(P) \mid P \in \mathcal{P}\}$. To see this, observe that

$$\begin{aligned}(x_1, \ldots, x_n) \in Z(P_1) \cup Z(P_2) &\Leftrightarrow P_1(x_1, \ldots, x_n) = 0 \text{ or } P_2(x_1, \ldots, x_n) = 0 \\ &\Leftrightarrow P_1(x_1, \ldots, x_n) \cdot P_2(x_1, \ldots, x_n) = 0 \\ &\Leftrightarrow (x_1, \ldots, x_n) \in Z(P_1P_2).\end{aligned}$$

Hence, $Z(P_1) \cup Z(P_2)$ is the intersection of a subfamily of $\{Z(P) \mid P \in \mathcal{P}\}$ (namely, $\{Z(P_1P_2)\}$ itself). □

30 Problem 13D3

For a polynomial P in n real variables, let $Z(P) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid P(x_1, \ldots, x_n) = 0\}$. Let $\mathcal{P}$ be the collection of all such polynomials.

Definition 30.1. The Zariski topology on $\mathbb{R}^n$ is the one having the set $\{Z(P) \mid P \in \mathcal{P}\}$ as a base for its closed sets.

Proposition 30.2. *On $\mathbb{R}$, the Zariski topology coincides with the cofinite topology.*

Proof. Let $\mathcal{F}_Z$ denote the closed sets in the Zariski topology and let $\mathcal{F}_{co}$ denote the closed sets in the cofinite topology. We show that $\mathcal{F}_Z = \mathcal{F}_{co}$.

Let $F \in \mathcal{F}_Z$. Observe that $F = \bigcap Z(P_\alpha)$ for some subfamily $\{P_\alpha\} \subset \mathcal{P}$. If the subfamily is merely the zero polynomial, then every real number is a root, and so $F = \mathbb{R}$, which belongs to $\mathcal{F}_{co}$. Otherwise, the number of roots of each polynomial is finite, and so the intersection of these sets of roots is finite. Hence, F is finite, and so $F \in \mathcal{F}_{co}$.

Let $F \in \mathcal{F}_{co}$. If $F = \mathbb{R}$, then F corresponds to the set of roots the zero polynomial. If $F = \emptyset$, then F corresponds to the set of roots of a nonzero, constant polynomial. Otherwise, construct the polynomial $P(x) = \prod_{a\in F}(x - a)$ (since F is finite, this is indeed a polynomial). We see that F corresponds to the set of roots of the polynomial P. In any case, we conclude that $F \in \mathcal{F}_Z$. □

Proposition 30.3. *On $\mathbb{R}^n$ with $n \geq 2$, the Zariski topology does not coincide with the cofinite topology.*

Proof. Consider the polynomial $P(x_1, \ldots, x_n) = x_1$ for any $n \geq 2$. We see that $Z(P) = \{(0, y_2, \ldots, y_n) \mid y_i \in \mathbb{R}$ for all $2 \leq i \leq n\}$, which belongs to $\mathcal{F}_Z$. It does not belong to $\mathcal{F}_{co}$, however, as it is neither finite nor equal to $\mathbb{R}^n$. □

31 Problem 13E2

Definition 31.1. We say that a is an accumulation point of a set A in a space X provided each neighborhood of a meets A in some point other than a. We say a is a condensation point of A provided each neighborhood of a meets A in uncountably-many points. Let A' denote the set of accumulation points of A and $A^\bullet$ denote the set of condensation points of A.

Proposition 31.2. *For any subset A of a T_1 space, A' is a closed set.*

Proof. Let $x \notin A'$. There exists a basic neighborhood U of x such that $U \cap A \subset \{x\}$. In particular, this implies that there is an open set $O \subset U$ containing x such that $O \cap A \subset \{x\}$. Now, let $y \in O \setminus \{x\}$. In a T_1 space, singletons are closed, and so $O \setminus \{x\}$ is open. As $O \setminus \{x\}$ is an open neighborhood of y missing A, it follows that $y \notin A'$. As $x \notin A'$ by definition, we have that $O \cap A' = \emptyset$. Hence, A' is closed. □

Remark 31.3. If the space is merely T_0, the above proposition can fail. For a counterexample, consider $\mathbb{R}$ with the topology whose base is $\{(-\infty, a) \mid a \in \mathbb{R}\}$, which is a T_0 space but not a T_1 space (given $x, y \in \mathbb{R}$, we can find a neighborhood containing the smaller of x and y and excluding the other, but not vice versa). In this case, the derived set of $\{0\}$ is $(0, \infty)$, which is not closed.

Proposition 31.4. *For any subset A of a topological space, $A^\bullet$ is a closed set with $A^\bullet \subset A'$.*

Proof. Let $x \notin A^\bullet$. There exists a basic neighborhood U of x such that $U \cap A$ is a countable set. Now, for any $y \in U$, there is a neighborhood of y (namely, U itself) that meets A in only countably-many points. Hence, $y \notin A^\bullet$ for all $y \in U$. In other words, U is a neighborhood of x missing $A^\bullet$, and so $A^\bullet$ is closed.

Obviously, if every neighborhood of some point meets A in uncountably-many points, then it meets A at some point other than itself. Hence, $A^\bullet \subset A'$. □

32 Problem 13E4

Proposition 32.1. *Given a set A, let $A^1 = A'$, $A^2 = (A^1)')$, and so on. For any positive integer n, there is a set $A \subset \mathbb{R}$ such that $A, A^1, \ldots, A^{n-1}$ are nonempty and $A^n = \emptyset$.*

Proof. Consider $\mathbb{R}$ with the usual topology. The set $A = \{0\}$ has $A^1 = \emptyset$. For $n \geq 2$, define

$$A = \left\{ \left(\frac{1}{m_1}, \frac{1}{m_2}, \ldots, \frac{1}{m_{n-1}} \right) \mid m_i \in \mathbb{N} \text{ for all } i \right\}.$$

Observe that

$$\left\{ \left(\frac{1}{m_1}, \frac{1}{m_2}, \ldots, \frac{1}{m_{n-3}}, \frac{1}{m_{n-2}}, 0 \right) \mid m_i \in \mathbb{N} \text{ for all } i \right\} \subset A^1,$$

$$\left\{ \left(\frac{1}{m_1}, \frac{1}{m_2}, \ldots, \frac{1}{m_{n-3}}, 0, 0 \right) \mid m_i \in \mathbb{N} \text{ for all } i \right\} \subset A^2,$$

and so on. By induction, $A^k \neq \emptyset$ for all $k < n$, while $A^n = \emptyset$. □

33 Definitions

A space X is called Lindelöff if, whenever $\mathcal{G}$ is a collection of open sets such that $\bigcup\{G \mid G \in \mathcal{G}\} = X$ (such a collection is called an open cover), there is a countable subcollection $\{G_n \mid n \in \mathbb{N}\}$ such that $\bigcup\{G_n \mid n \in \mathbb{N}\} = X$.

Call a space X D_2 if distinct points can be put into disjoint clopen (i.e. simultaneously closed and open) sets and D_3 if it is T_0 and has a base of clopen sets.

34 Problem 1

Proposition 34.1. *Every D_3 space is D_2 and Tychonoff.*

Proof. Let x and y be elements of a D_3 space X. Given distinct $x, y \in X$, we can find a basic open (hence, clopen) neighborhood V containing one but not the other. Without loss of generality, let it be that $x \in V$ and $y \notin V$. As V is closed and $y \notin V$, there is a basic open (hence, clopen) neighborhood W that contains y and is disjoint from V. Hence, X is D_2.

To show that X is Tychonoff, we show that it is both T_1 and completely regular. Evidently, D_2 implies T_2, which in turn implies T_1. Now, let A be a closed set in X and let $x \notin A$. We can find disjoint open sets B and V_x such that $A \subset B$ and $x \in V_x$. Define the function $f : B \cup V_x \to \mathbf{I}$ sending B to 0 and V_x to 1. Since B and V_x are disjoint, the inverse image of any subset of $\mathbf{I}$ is either $\emptyset$, B, V_x, X, all of which are open. Hence, f is continuous, and so X is Tychonoff. □

35 Problem 2

Proposition 35.1. *Let $\mathcal{G}$ be an open cover of a D_3, Lindelöff space X. There is a (countable) partition $\mathcal{P}$ of X into clopen sets such that, for each $P \in \mathcal{P}$, there is $G \in \mathcal{G}$ such that $P \subset G$.*

Proof. For all $G \in \mathcal{G}$, G can be represented as a union of basic neighborhoods containing each of its points. Hence, we can construct a clopen cover $\mathcal{G}'$ of X. Since X is Lindelöff, we can assume that $\mathcal{G}'$ is countable. That is, $\mathcal{G}' = \{G_n \mid n \in \mathbb{N}\}$.

Now, for each $n \in \mathbb{N}$, define $P_n = G_n \setminus \bigcup_{i=1}^{n-1} G_i$ and let $\mathcal{P} = \{P_n \mid n \in \mathbb{N}\}$. Evidently, $\mathcal{P}$ is countable with $\bigcup\{P \mid P \in \mathcal{P}\} = X$ and $P_i \cap P_j = \emptyset$ whenever $i \neq j$. Furthermore, for all $n \in \mathbb{N}$, $P_n \subset G_n \subset G$ for some $G \in mathcalG$ (since the G_n were chosen to be basic clopen subsets of the open sets of the cover $\mathcal{G}$). □

Remark 35.2. Intuitively, the above proposition says we can chop X up into small enough clopen pieces so that each piece fits inside some member of $\mathcal{G}$.