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Math 704 Homework

Problem 1

Prove that the Cantor set $\mathcal{C}$ is totally disconnected and perfect. In other words, given two distinct points $x, y \in \mathcal{C}$, there is a point $z \notin \mathcal{C}$ that lies between $x$ and $y$, and yet $\mathcal{C}$ has no isolated points.

*Proof.* Let $x, y \in \mathcal{C}$ be distinct. Then, $x, y \in \mathcal{C}_k$ for all $k \in \mathbb{N}$. Now, since $x$ and $y$ are distinct, we can find $N \in \mathbb{N}$ such that $\frac{1}{3^N} < |x-y|$. Hence, $x$ and $y$ belong to different intervals of $\mathcal{C}_N$. By the construction of the Cantor set, there must be at least one interval between $x$ and $y$ which does not belong to $\mathcal{C}_N$, and so does not belong to $\mathcal{C}$. Select one such interval. Choosing any point $z$ in this interval satisfies that $z$ lies between $x$ and $y$ and $z \notin \mathcal{C}$. Therefore, $\mathcal{C}$ is totally disconnected.

To see that $\mathcal{C}$ is perfect, let $\epsilon > 0$ be given and consider $B(x, \epsilon)$ for any $x \in \mathcal{C}$. Let $I_k$ denote the interval to which $x$ belongs in $\mathcal{C}_k$. We can find $N \in \mathbb{N}$ such that $I_N \subset B(x, \epsilon)$. Now, this interval must have two endpoints $a_N$ and $b_N$ (one of which could possibly be equal to $x$). By the construction of the Cantor set, we know that the endpoints of any interval are never removed, and so $a_N, b_N \in \mathcal{C}$. Furthermore, we have that $a_N, b_N \in I_N \subset B(x, \epsilon)$. Therefore, $x$ is not isolated. □

Problem 2

The Cantor set $\mathcal{C}$ can also be described in terms of ternary expansions.

a) Every number in $[0, 1]$ has a ternary expansion

$$x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}, \text{ where } a_k = 0, 1, \text{ or } 2.$$

Prove that $x \in \mathcal{C}$ if and only if $x$ has a representation as above where every $a_k$ is either 0 or 2.

*Proof.* ($\Rightarrow$) Let $x \in \mathcal{C}$. We build a ternary expansion for $x$ of the desired form as follows.Consider $\mathcal{C}_1$. It must be that $x$ belongs to one of $[0, \frac{1}{3}]$ (in which case let first digit of the ternary expansion for $x$ be 0) or $[\frac{2}{3}, 1]$ (in which case let first digit of the ternary expansion for $x$ be 2). Next, consider $\mathcal{C}_2$. The interval of $\mathcal{C}_1$ to which $x$ currently belongs will be divided into three subintervals, and so we append a 0 to the ternary expansion of $x$ if it belongs to the leftmost subinterval or a 2 if it belongs to the rightmost subinterval. Continuing in this way, we see that $x$ has an associated ternary expansion containing only the digits 0 and 2.

($\Leftarrow$) Let

$$x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}, \text{ where } a_k = 0 \text{ or } 2.$$

We can locate $x$ on the real line as follows. If $a_1 = 0$, we choose the left subinterval of $\mathcal{C}_1$. If $a_1 = 2$, we choose the rightmost subinterval of $\mathcal{C}_1$. When we form $\mathcal{C}_2$, the interval we have just chosen will be subdivided into three subintervals. If $a_2 = 0$, we select the leftmost subinterval. If $a_2 = 2$, we select the rightmost subinterval. Continue in this way. Since the length of these intervals can be made arbitrarily small, we see that the ternary expansion of $x$ uniquely specifies its location on the real line. □

b) The Cantor-Lebesgue function is defined on $\mathcal{C}$ by

$$F(x) = \sum_{k-1}^{\infty} \frac{b_k}{2^k} \text{ if } x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}, \text{ where } b_k = \tfrac{a_k}{2}$$

In this definition, we choose the expansion of $x$ in which $a_k = 0$ or 2. Show that $F$ is well-defined and continuous on $\mathcal{C}$, and moreover $F(0) = 0$ as well as $F(1) = 1$.

*Proof.* Let $x, x' \in \mathcal{C}$ with $x = x'$. Denote the $k$th digit of the ternery expansion of $x$ and $x'$ by $a_k$ and $a'_k$, respectively.

Claim $a_k = a'_k$ for all $k$.
Proof of Claim Suppose not. Then, $a_N \neq a'_N$ for some $N$. From the construction in part (a), we see that $x$ and $x'$ must belong to different subintervals in $\mathcal{C}_N$, and so $x \neq x'$, which is a contradiction.

Now, let $b_k = \frac{a_k}{2}$ and $b'_k = \frac{a'_k}{2}$. Then $b_k = b'_k$ for all $k$. Hence

$$F(x) = \sum_{k=1}^{\infty} \frac{b_k}{2^k} = \sum_{k=1}^{\infty} \frac{b'_k}{2^k} = F(x')$$

and so $F$ is well-defined.

To see that $F$ is continuous, let $\epsilon > 0$ be given and $x, x' \in \mathcal{C}$ so that $|F(x) - F(x')| < \epsilon$. Consider the binary expansion of $\epsilon$ (denote the $k$th digit of $\epsilon$ by $\epsilon_k$). Construct $\delta > 0$ such that $\delta_k = 2\epsilon_k$ for all $k$. Let $N$ be the first nonzero digit of $\delta$ and $\epsilon$. Then, $|x - x'| < \delta$ implies that the first $N-1$ digits of $x$ and $x'$ agree. Hence, the first $N-1$ digits of $F(x)$ and $F(x')$ agree, and so $|F(x) - F(x)'| < \epsilon$. Therefore, $F$ is continuous.

By the construction in part (a), we know that 0 is represented in ternary form by always choosing the leftmost subinterval, and so for $x = 0$, $b_k = \frac{0}{2} = 0$ for all $k$. Similarly, 1 is represented in ternary form by always chosing the rightmost subinterval, and so for $x = 1$, $b_k = \frac{2}{2} = 1$ for all $k$. Hence

$$F(0) = \sum_{k=1}^{\infty} \frac{0}{2^k} = 0$$
$$F(1) = \sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

□

c) Prove that $F : \mathcal{C} \to [0, 1]$ is surjective.

*Proof.* Let $y \in [0, 1]$. Then $y$ has a corresponding binary expansion. Let $b_k$ denote the $k$th digit of this expansion. Construct a string $s$ such that $s_k = 2b_k$ for all $k$, where $s_k$ denotes the $k$th digit of $s$. This construction uniquely identifies some ternary string using only 0s and 2s. From part (a), we know that $s$ corresponds uniquely to some $x \in \mathcal{C}$. Now, it is clear from our construction of $x$ that $F(x) = y$. □

Problem 3

Recall that every open set in $\mathbb{R}$ is the disjoint union of open intervals. The analogue in $\mathbb{R}^d$, $d \geq 2$ is generally false. Prove the following:

a) An open disc in $\mathbb{R}^2$ is not the disjoint union of open rectangles.

*Proof.* Suppose, to the contrary, an open disc $\mathcal{O} \subset \mathbb{R}^2$ is the disjoint union of open rectangles. Choose some open rectangle $R_1 \in \mathcal{O}$ and let $x \in \delta R_1$. Then, for all $\epsilon > 0$, $B(x, \epsilon) \cap R_1 \neq \phi$ and $B(x, \epsilon) \cap R_1^c \neq \phi$. Hence, $x \notin R_1$, and so there must be an open rectangle $R_2 \in \mathcal{O}$ with $x \in R_2$. This implies that there is $\epsilon_0 > 0$ such that $B(x, \epsilon_0) \subset R_2$. By our previous observation, $B(x, \epsilon_0) \cap R_1 \neq \phi$. Taken together, we see that $R_1 \cap R_2 \neq \phi$, which is a contradiction with the fact that $\mathcal{O}$ is the disjoint union of open rectangles. □

b) An open connected set $\Omega$ is the disjoint union of open rectangles if and only if $\Omega$ is itself an open rectangle.

*Proof.* ($\Rightarrow$) Let $\Omega$ be the disjoint union of open rectangles. Suppose, to the contrary, that $\Omega$ is not itself an open rectangle. Then, $\Omega$ contains at least two open rectangles. By the argument in part (a), we see that these rectangles cannot be disjoint, which is a contradiction. Hence, it must be that $\Omega$ is itself an open rectangle.

($\Leftarrow$) Let $\Omega$ be an open rectangle. Then $\Omega$ is the disjoint union of a single open rectangle (namely, $\Omega$ itself). □

<u>Problem 1</u> (Cantor-like Sets)

Construct a closed set $\hat{\mathcal{C}}$ so that at the $k$th stage of the construction one removes $2^{k-1}$ centrally situated open intervals each of length $l_k$ with

$$l_1 + 2l_2 + \cdots + 2^{k-1}l_k < 1$$

a) If $l_j$ are chosen small enough, then

$$\sum_{k=1}^{\infty} 2^{k-1}l_k < 1$$

In this case, show that $m(\hat{\mathcal{C}}) > 0$, and in fact,

$$m(\hat{\mathcal{C}}) = 1 - \sum_{k=1}^{\infty} 2^{k-1}l_k$$

*Proof.* First, we claim that $\hat{\mathcal{C}}$ is measurable. Denote by $O_k$ the union of the open sets removed from $[0,1]$ at step $k$ of the construction. Since the union of an arbitrary number of open sets is open, each of the $O_k$ is open. Furthermore, $O = \bigcup_{k=1}^{\infty} O_k$ is open. Now, we have that $\hat{\mathcal{C}} = [0,1] \setminus O$ is closed and therefore measurable (as all closed sets are measurable).

Now, to determine $m(\hat{\mathcal{C}})$, observe that both $O$ and $\hat{\mathcal{C}}$ are measurable ($O$ is measurable because it is open) and disjoint and that $O \cup \hat{\mathcal{C}} = [0,1]$. Then

$$\begin{aligned} m([0,1]) &= m(O) + m(\hat{\mathcal{C}}) \\ m(\hat{\mathcal{C}}) &= m([0,1]) - m(O) \end{aligned}$$

Furthermore observe that all of the $O_k$ are open (and so measurable) and disjoint with $O = \bigcup_{k=1}^{\infty} O_k$. If we further break the $O_k$ into their constituent open subsets, these properties still hold. Hence

$$\begin{aligned} m(\hat{\mathcal{C}}) =& m([0,1]) - m(O) \\ =& m([0,1]) - \sum_{k=1}^{\infty} m(O_k) \\ =& 1 - \sum_{k=1}^{\infty} 2^{k-1}l_k \end{aligned}$$

□

b) Show that if $x \in \hat{\mathcal{C}}$, then there exists a sequence of points $\{x_n\}_{n=1}^{\infty}$ such that $x_n \notin \hat{\mathcal{C}}$, yet $x_n \to x$ and $x_n \in I_n$, where $I_n$ is a sub-interval in the complement of $\hat{\mathcal{C}}$ with $|I_n| \to 0$.

*Proof.* Observe first that since $\sum_{k=1}^{\infty} 2^{k-1}l_k < 1$, the tail of the series must go to zero. That is, for any $\epsilon > 0$, there exists $N$ such that $l_n < \epsilon$ for all $n \geq N$. Now, let $x \in \hat{\mathcal{C}}$. Let $\hat{\mathcal{C}}_k$ denote the $k$ stage of the construction. For each $k$, $x$ belongs to some closed subset $S_k$ of $C_k$. Let $I_k$ be the open interval removed from $S_k$ to proceed to the next step of the construction. We take any $x_k \in I_k$ to form our sequence $\{x_n\}_{n=1}^{\infty}$. Clearly, each $x_k$ belongs to an sub-interval in the complement of $\hat{\mathcal{C}}$. Furthermore, $|I_k| = l_k \to 0$. It remains to show that $x_n \to x$.

From the construction of $\hat{\mathcal{C}}_k$ and our selection of $x_n$, it is clear that

$$|x - x_n| < |I_n| + |S_n|$$

By our previous observation, we know that $|I_n| = l_n \to 0$. Now

$$\begin{aligned} |S_n| &= \frac{1 - \sum_{k=1}^{n} 2^{k-1} l_k}{2^n} \\ &\le \frac{1}{2^n} \\ &\to 0 \text{ as } n \to \infty \end{aligned}$$

Hence, $|x - x_n| \to 0$. That is, $\{x_n\}_{n=1}^{\infty}$ converges to $x$. □

c) Prove as a consequence that $\hat{\mathcal{C}}$ is perfect and contains no open interval.

*Proof.* To see that $\hat{\mathcal{C}}$ is perfect, let $\epsilon > 0$ be given and consider $B(x, \epsilon)$ for any $x \in \hat{\mathcal{C}}$. We can find $N \in \mathbb{N}$ such that $S_N \subset B(x, \epsilon)$. Now, this interval must have two endpoints $a_N$ and $b_N$ (one of which could possibly be equal to $x$). By the construction of $\hat{\mathcal{C}}$, we know that the endpoints of any interval are never removed, and so $a_N, b_N \in \hat{\mathcal{C}}$. Furthermore, we have that $a_N, b_N \in S_N \subset B(x, \epsilon)$. Therefore, $x$ is not isolated.

Suppose, to the contrary, that there exists an open interval $O \in \hat{\mathcal{C}}$. Then, for any $x \in O$, there exists $\epsilon_0$ such that $B(x, \epsilon_0) \subseteq O$. Let $\epsilon < \epsilon_0$. Then, there can be no sequence $\{x_n\}_{n=1}^{\infty}$ of the type described in part (b) whose limit is $x$, since $B(x, \epsilon_0) \subseteq \hat{\mathcal{C}}$ implies that $|x - x_n| > \epsilon_0 > \epsilon$ for all $n$. This contradicts the conclusion of part (b), and so it must be that $\hat{\mathcal{C}}$ contains no open interval. □

d) Show also that $\hat{\mathcal{C}}$ is uncountable.

*Proof.* We claim that $\hat{\mathcal{C}}$ is in one-to-one correspondence with infinite ternary strings containing only 0s and 2s, and so is uncountable.

($\Rightarrow$) Let $x \in \hat{\mathcal{C}}$. We build a ternary string for $x$ of the desired form as follows. Consider $\hat{\mathcal{C}}_1$. When we remove the centrally situated open interval, it must be that $x$ belongs to either the left closed subinterval (in which case let first digit of the ternary string for $x$ be 0) or the right closed subinterval (in which case let first digit of the ternary string for $x$ be 2). Next, consider $\hat{\mathcal{C}}_2$. The interval of $\hat{\mathcal{C}}_1$ to which $x$ currently belongs will be divided into three subintervals, and so we append a 0 to the ternary string for $x$ if it belongs to the leftmost subinterval or a 2 if it belongs to the rightmost subinterval. Continuing in this way, we see that $x$ has an associated ternary string containing only the digits 0 and 2.

($\Leftarrow$) Let $s$ be an infinite ternary string containing only 0s and 2s. We associate can with $s$ an $x \in \hat{\mathcal{C}}$ as follows. If the first digit of $s$ is 0, we choose the left subinterval of $\hat{\mathcal{C}}_1$. If the first digit of $s$ is 2, we choose the rightmost subinterval of $\hat{\mathcal{C}}_1$. When we form $\hat{\mathcal{C}}_2$, the interval we have just chosen will be subdivided into three subintervals. If the second digit of $s$ is 0, we select the leftmost subinterval. If the second digit of $s$ is 2, we select the rightmost subinterval. Continue in this way. Since each $x \in \hat{\mathcal{C}}$ belongs to a singleton set, we see that $s$ will specify some $x \in \hat{\mathcal{C}}$. □

<u>Problem 2</u>

Suppose $E$ is a given set and $\mathcal{O}_n$ is the open set

$$\mathcal{O}_n = \{x : d(x, E) < \tfrac{1}{n}\}$$

a) Show that if $E$ is compact, then $m(E) = \lim_{n\to\infty} m(\mathcal{O}_n)$.

*Proof.* By the Heine-Borel theorem, $E$ is closed and bounded. Now, since $E$ is closed, $E$ is measurable. Now, we want to apply the following fact

$$\text{If } \mathcal{O}_k \searrow E \text{ and } m(\mathcal{O}_k) < \infty \text{ for some } k, \text{ then } m(E) = \lim_{n\to\infty} m(\mathcal{O}_n)$$

It is clear that $\mathcal{O}_k \supseteq \mathcal{O}_{k+1}$ for all $k$ (if $d(x, E) < \frac{1}{k+1}$, then certainly $d(x, E) < \frac{1}{k}$). We show next that $E = \bigcap_{k=1}^{\infty} \mathcal{O}_k$.

Let $x \in E$. Then, $d(x, E) = 0 < \frac{1}{k}$ for all $k$. Hence, $x \in \mathcal{O}_k$ for all $k$, and therefore $x \in \bigcap_{k=1}^{\infty} \mathcal{O}_k$. That is, $E \subseteq \bigcap_{k=1}^{\infty} \mathcal{O}_k$.

Let $x \in \bigcap_{k=1}^{\infty} \mathcal{O}_k$. Then, $d(x, E) < \frac{1}{k}$ for all $k$. That is, $d(x, E) \to 0$. Now, since $E$ is compact, $d(x, E)$ attains its minimum, and so $d(x, E) = 0$. This implies that $x \in E$, and therefore $\bigcap_{k=1}^{\infty} \mathcal{O}_k \subseteq E$.

To see that $m(\mathcal{O}_k) < \infty$ for some $k$, let $N$ be any fixed natural number. Since $E$ is bounded, there exists $x \in E$ and $0 < r < \infty$ such that $E \subseteq B(x, r)$. Now, let $y \in \mathcal{O}_N$. It follows that

$$\begin{aligned} d(x, y) &\leq r + d(y, E) \\ &< r + \frac{1}{N} \end{aligned}$$

and so $y \in B(x, r + \frac{1}{N})$. Therefore, $\mathcal{O}_N \subseteq B(x, r + \frac{1}{N})$. Now, since $B(x, r + \frac{1}{N})$ is an open ball of finite radius, it is measurable with finite measure. Since $\mathcal{O}_N$ is open with $\mathcal{O}_N \subseteq B(x, r + \frac{1}{N})$, $\mathcal{O}_N$ is measurable with finite measure.

Having satisfied the hypotheses of the aforementioned fact, we conclude that $m(E) = \lim_{n\to\infty} m(\mathcal{O}_n)$. □

b) Show that the conclusion in (a) may be false for $E$ closed and unbounded or $E$ open and bounded.

*Proof.* Let $E = \mathbb{Z}$ (which is closed and unbounded) in $\mathbb{R}$. Since $\mathbb{Z}$ is a collection of singleton points, $m(\mathbb{Z}) = 0$. On the other hand, each $O_n$ is the union of countably infinitely many open balls of radius $\frac{1}{n}$, and so $m(O_n)$ is infinite for all $n$. Hence, $\lim_{n\to\infty} m(O_n) = \infty$.

Let $\{r_1, \dots, r_n, \dots\} = \mathbb{Q} \cap (0, 1)$. Let $E_n = (r_n - \frac{\epsilon}{2^{n+1}}, r_n + \frac{\epsilon}{2^{n+1}})$ for all $n$. Finally, let

$$E = (\bigcup_{n=1}^{\infty} E_n) \bigcap (0, 1)$$

(which is open and bounded). Now, we have

$$O_n = \bigcup_{k=1}^{\infty} (r_k - \frac{\epsilon}{2^{k+1}} - \frac{1}{n}, r_k + \frac{\epsilon}{2^{k+1}} + \frac{1}{n})$$

Now, $O_n$ is an open cover of $(0, 1)$ for all $n$ since, for any $x \in (0, 1)$, we can find a rational number $r_k$ within $\frac{1}{n}$ of $x$, and so $x \in B(r_k, \frac{1}{n}) \subseteq O_n$. Hence, $m(O_n) \geq 1$ for all $n$. Therefore, for $\epsilon < 1$,

$$m(E) \leq \epsilon < 1 \leq \lim_{n\to\infty} m(O_n)$$

□

<u>Problem 3</u>

Let $A$ be the subset of $[0, 1]$ which consists of all numbers which do not have the digit 4 appearing in their decimal expansion. Find $m(A)$.

*Proof.* Let $x \in A$. Given any $\epsilon > 0$, we can find $N$ such that $\frac{a}{10^N} < \epsilon$ for all $a \in \{1, 2, \dots, 9\}$. That is, $x + \frac{a}{10^N} \in B(x, \epsilon)$ for all $a$. Hence, we can choose $a$ so that the $N$th digit in the decimal expansion of $x + \frac{a}{10^N}$ is 4. Therefore, $A$ is the uncountable union of disjoint singleton sets, and so $m(A) = 0$. □

Problem 4 (The Borel-Cantelli Lemma)

Suppose $\{E_k\}_{k=1}^{\infty}$ is a countable family of measurable subsets of $\mathbb{R}^d$ and that

$$\sum_{k=1}^{\infty} m(E_k) < \infty$$

Let

$$\begin{aligned} E &= \{x \in \mathbb{R}^d \mid x \in E_k, \text{ for infinitely-many } k\} \\ &= \overline{\lim}_{k\to\infty}(E_k) \end{aligned}$$

a) Show that $E$ is measurable.

*Proof.* Observe first that

$$E = \bigcap_{n=1}^{\infty} \bigcup_{k\geq n} E_k$$

since, if $x \in E_k$ for infinitely many $k$, it will appear in the union for all $k$, and will be included in the countable intersection. Now, since each $E_k$ is measurable, $\bigcup_{k\geq n} E_k$ is measurable for all $n$ (since the countable union of countable sets is countable). This implies that $E = \bigcap_{n=1}^{\infty} \bigcup_{k\geq n} E_k$ is countable (since the countable intersection of measurable sets is measurable). □

b) Prove $m(E) = 0$.

*Proof.* Since $\sum_{k=1}^{\infty} m(E_k) < \infty$, it must be that the tail of the summation goes to 0. That is, for any $\epsilon > 0$, there exists $N$ such that

$$\sum_{k=N}^{\infty} m(E_k) < \epsilon$$

Now,

$$E = \bigcap_{n=1}^{\infty} \bigcup_{k\geq n} E_k \subseteq \bigcup_{k=N}^{\infty} E_k$$

and so we conclude

$$\begin{aligned} \epsilon &> \sum_{k=N}^{\infty} m(E_k) \\ &\geq m(\bigcup_{k=N}^{\infty} E_k) \\ &\geq m(\bigcap_{n=1}^{\infty} \bigcup_{k\geq n} E_k) \\ &= m(E) \end{aligned}$$

and so $m(E) = 0$. □

Problem 1

Show that there exist closed sets $A$ and $B$ with $m(A) = m(B) = 0$, but $m(A + B) > 0$:

a. In $\mathbb{R}$, let $A = \mathcal{C}$ (the Cantor set), $B = \frac{\mathcal{C}}{2}$.

*Proof.* Recall that the Cantor set (here, $A$) contains no open interval, and so has measure 0. Similarly, $B$ has measure 0. Recall also that an element belongs to $A$ if and only if it has a ternary expansion using only 0s and 2s. Hence, an element belongs to $B$ if and only if it has a ternary expansion using only 0s and 1s. Now, let $x \in [0,1]$. We choose $a \in A$ and $b \in B$ such that $x = a + b$ (and so show that $x \in A + B$) as follows:

If the $k$th digit of $x$ is 0, specify that the $k$th digit of $a$ is 0 and the $k$th digit of $b$ is 0.
If the $k$th digit of $x$ is 1, specify that the $k$th digit of $a$ is 0 and the $k$th digit of $b$ is 1.
If the $k$th digit of $x$ is 2, specify that the $k$th digit of $a$ is 2 and the $k$th digit of $b$ is 0.

Hence, $A + B \supset [0,1]$, and so by problem 4a, we have that $A + B$ is measurable (since $A + B$ contains a subset of nonzero measure). Furthermore, $m(A + B) \geq m([0,1]) = 1$. □

b. In $\mathbb{R}^2$, observe that if $A = I \times \{0\}$ and $B = \{0\} \times I$ (where $I = [0,1]$), then $A + B = I \times I$.

*Proof.* Lines in $\mathbb{R}^2$ have measure 0, since, in the limit, a covering a closed cubes will have sides of length 0. Hence, $A$ and $B$ both have measure 0. Now, consider

$$I \times I = \{(a,b) \mid a, b \in [0,1]\}$$

This is a closed cube with sides of length 1, and is therefore measurable with measure 1. □

Problem 2

Prove that there is a continuous function that maps a Lebesgue measurable set to a non-measurable set.

*Proof.* Consider the function $F : \mathcal{C} \to [0,1]$ defined previously. We have already shown that $F$ is continuous and surjective. Now, let $N$ be the non-measurable set described in the text, and let $\mathcal{C}'$ denote the preimage of $N$ under $F$. Since $F$ is surjective, we know that $\mathcal{C}'$ is nonempty. Furthermore, $\mathcal{C}' \subseteq \mathcal{C}$ since $N \subseteq [0,1]$. Now, consider the function

$$G : \mathcal{C}' \to N$$
$$G(x) = F(x) \text{ for all } x \in \mathcal{C}'$$

(i.e. the function $F$ restricted to the domain $\mathcal{C}'$). $G$ is surjective by definition of $\mathcal{C}'$. Furthermore, since $F$ is continuous, $G$ is continuous. Finally, we see that $m(\mathcal{C}') \leq m(\mathcal{C}) = 0$, since $\mathcal{C}' \subseteq \mathcal{C}$. Therefore, $G$ is a continuous function mapping a Lebesgue measurable set onto a non-measurable set. □

Problem 3

Let $E$ be a subset of $\mathbb{R}$ with $m^*(E) > 0$. Prove that for each $0 < \alpha < 1$, there exists an open interval $I$ so that

$$m^*(E \cap I) \geq \alpha m^*(I)$$

*Proof.* Choose $\mathcal{O} \supset E$ such that $m^*(E) \geq \alpha m^*(\mathcal{O})$ (this can always be done). We can write

$$\mathcal{O} = \bigcup_{i=1}^{\infty} \mathcal{O}_i,\ \mathcal{O}_i \text{ open and disjoint}$$

and hence

$$\begin{aligned} E &= E \cap \mathcal{O} \\ &= E \cap \bigcup_{i=1}^{\infty} \mathcal{O}_i \\ &= \bigcup_{i=1}^{\infty} (E \cap \mathcal{O}_i) \end{aligned}$$

Now, suppose that $m^*(E \cap \mathcal{O}_i) < \alpha m^*(\mathcal{O}_i)$ for all $i$. Then

$$\begin{aligned} m^*(E) &= m^*(\bigcup_{i=1}^{\infty}(E \cap \mathcal{O}_i)) \\ &= \sum_{i=1}^{\infty} m^*(E \cap \mathcal{O}_i) && \text{(by the disjointness of the } E \cap \mathcal{O}_i) \\ &< \alpha \sum_{i=1}^{\infty} m^*(\mathcal{O}_i) && \text{(by hypothesis)} \\ &= \alpha m^*(\mathcal{O}) && \text{(by disjointness of the } \mathcal{O}_i) \end{aligned}$$

which is a contradiction with the fact that $m^*(E) \geq \alpha m^*(\mathcal{O})$. Hence, it must be that, for some $k$, $m^*(E \cap \mathcal{O}_k) \geq \alpha m^*(\mathcal{O}_k)$, which proves the claim. □

Problem 4

Let $\mathcal{N}$ denote the non-measurable subset of $I = [0, 1]$ constructed at the end of Section 3.

a. Prove that if $E$ is a measurable subset of $\mathcal{N}$, then $m(E) = 0$.

*Proof.* Let $\{r_k\}_{k=1}^{\infty}$ be an enumeration of the rationals in the interval $[-1, 1]$ and let $E_k = E + r_k$ for each $k$. Since $E \subseteq N$, $E_k \subseteq N_k$ for each $k$. Since each of the $N_k$ are pairwise disjoint, each of the $E_k$ are pairwise disjoint. Now, the Lebesgue measure is translation invariant, so $m(E_k) = m(E)$ for each $k$. We also have that $\bigcup_{k=1}^{\infty} E_k \subseteq \bigcup_{k=1}^{\infty} N_k \subseteq [-1, 2]$. It follows that

$$\begin{aligned} \sum_{k=1}^{\infty} m(E_k) &= m(\bigcup_{k=1}^{\infty} E_k) && \text{(by the disjointness of the } E_k) \\ &\leq 3 && \text{(since } \bigcup_{k=1}^{\infty} E_k \subseteq [-1, 2]) \end{aligned}$$

But $m(E_k) = m(E)$ for each $k$. Hence

$$\begin{aligned} 3 &\geq \sum_{k=1}^{\infty} m(E_k) \\ &= \sum_{k=1}^{\infty} m(E) \end{aligned}$$

which implies that $m(E) = 0$. □

b. If $G$ is a subset of $\mathbb{R}$ with $m^*(G) > 0$, prove that a subset of $G$ is non-measurable.

*Proof.* Since $m(G) > 0$, we can find for any $\epsilon > 0$ a closed interval $[a, b] \subseteq G$ with $m(G \setminus [a, b]) \leq \epsilon$. Now, consider the set $G - a$ ($G$ translated by $-a$ units). Since the Lebesgue measure is translation invariant, $m(G - a) = m(G)$. Furthermore, the interval $[0, b - a] \subseteq G - a$. Let $A = [0, b - a] \cap N$. Observe that $A \subseteq G$. Suppose $A$ is measurable. Since $A \subseteq N$, $m(A) = 0$ by part (a). It follows that

$$\begin{aligned} m(G) &= m(G - a) \\ &= m((G - a) \setminus A) + m(A) \\ &\leq \epsilon + 0 \\ &= \epsilon \end{aligned}$$

Since $\epsilon$ can be chosen arbitrarily small, we conclude that $m(G) = 0$, which is a contradiction. Hence, it must be that $A$ is non-measurable. □

Problem 1

Let $\{f_n\}$ be a sequence of measurable functions on $[0,1]$ with $|f_n(x)| < \infty$ for almost every $x$. Show that there exists a sequence $c_n$ of positive real numbers such that

$$\frac{f_n(x)}{c_n} \to 0 \text{ for almost every } x.$$

*Proof.* For a fixed value of $n$, define the set $F_k$ to be $\{x \in [0,1] : |f_n(x)| \geq k\}$ (which are measurable, since $f_n$ measurable). We see that $F_1 \supseteq F_2 \supseteq \cdots$ (as $k$ grows larger, $F_k$ can only get smaller). Now, $m(F_1) \leq 1 < \infty$, and so $\lim_{k\to\infty} m(F_k) = m(\bigcap_{k=0}^{\infty} F_k)$. Since $|f_n(x)| < \infty$ for almost every $x$, it must be that $m(\bigcap_{k=0}^{\infty} F_k) = 0$, which implies that there is some $k_n$ such that $m(F_{k_n}) < \frac{1}{2^n}$. In other words,

$$m(\{x \in [0,1] : |f_n(x)| \geq k_n\}) < \tfrac{1}{2^n}$$

or, equivalently

$$m(\{x \in [0,1] : \left|\tfrac{f_n(x)}{nk_n}\right| \geq \tfrac{1}{n}\}) < \tfrac{1}{2^n}$$

Now, for each $n$, define $E_n$ to be the set $\{x \in [0,1] : \left|\frac{f_n(x)}{nk_n}\right| \geq \frac{1}{n}\}$. We have now a countable family $\{E_n\}$ of measurable subsets of $\mathbb{R}$ with $\sum_{k=1}^{\infty} E_k < \infty$. Define $E$ to be the set $\{x \in \mathbb{R} : x \in E_k \text{ for infinitely many } k\}$. The Borel-Cantelli lemma implies $m(E) = 0$. In other words, the subset of $[0,1]$ whose image is nonzero under $\left|\frac{f_n(x)}{nk_n}\right|$ for infinitely-many $n$ has measure zero. Therefore,

$$\frac{f_n(x)}{nk^n} \to 0 \text{ for almost every } x$$

□

Problem 2

Let $\chi_{[0,1]}$ be the characteristic function of $[0,1]$. Show that there is no everywhere continuous function $f$ on $\mathbb{R}$ such that

$$f(x) = \chi_{[0,1]}(x) \text{ almost everywhere.}$$

*Proof.* Suppose $f(x) = \chi_{[0,1]}(x)$ almost everywhere. Consider the set $(-\frac{\delta}{2}, 0) \cup (0, \frac{\delta}{2})$ for any $\delta > 0$. Since $f(x) = \chi_{[0,1]}(x)$ almost everywhere, we can find $x_0 \in (-\frac{\delta}{2}, 0)$ such that $f(x_0) = 0$ and $x_1 \in (0, \frac{\delta}{2})$ such that $f(x_1) = 1$. Now, let $0 < \epsilon < 1$ be given. Regardless of how small we make $\delta$, we have

$$|x_0 - x_1| < \delta \text{ yet } |f(x_0) - f(x_1)| = |0 - 1| = 1 > \epsilon$$

Therefore, $f$ is not continuous everywhere. □

Problem 3

Let $\mathcal{N}$ denote the measurable set constructed in the text. Recall that measurable subsets of $\mathcal{N}$ have measure zero. Show that the set $\mathcal{N}^c = [0,1] \setminus \mathcal{N}$ satisfies $m^*(\mathcal{N}^c) = 1$, and conclude that

$$m^*(\mathcal{N}) + m^*(\mathcal{N}^c) \neq m^*(\mathcal{N} \cup \mathcal{N}^c)$$

although $\mathcal{N}$ and $\mathcal{N}^c$ are disjoint.

*Proof.* Suppose, to the contrary, that $0 \leq m^*(\mathcal{N}^c) < 1$. We can find a measurable set $U$ such that $N^c \subset U \subset [0,1]$ with $m(U) = 1 - \epsilon$ for some small $\epsilon$. Now, $U^c \subseteq \mathcal{N}$, and so $m(U^c) = 0$ (since measurable subsets of a non-measurable set have measure zero). We have that

$$\begin{aligned} 1 - \epsilon &= m(U) + m(U^c) \\ &= m(U \cup U^c) \\ &= m([0,1]) \\ &= 1 \end{aligned}$$

which is a contradiction. Hence, $m^*(\mathcal{N}^c) = 1$. Now, during the course of the proof that $\mathcal{N}$ is non-measurable, it was shown that $m^*(\mathcal{N}) > 0$. So, we have $m^*(\mathcal{N}) + m^*(\mathcal{N}^c) > 1$. On the other hand, we have that $m^*(\mathcal{N} \cup \mathcal{N}^c) = m^*([0,1]) = 1$. Therefore, $m^*(\mathcal{N}) + m^*(\mathcal{N}^c) \neq m^*(\mathcal{N} \cup \mathcal{N}^c)$.

□

Problem 4

Give an example of a measurable function $f$ and a continuous function $\Phi$ so that $f \circ \Phi$ is non-measurable.

(Hint: Let $F$ be as in problem 2d and define $G : [0,1] \to [0,2]$ by $G(x) = x + F(x)$. Check that $G$ is strictly increasing, continuous and onto. Take $C_2$ equal to the standard Cantor set and $C_1 = G(C_2)$. Show $m(C_1) = 1$. Now let $\Phi = G^{-1}$. Let $N \subset C_1$ be non-measurable, and take $f = \chi_{\Phi(N)}$.)

Use the construction in the hint to show that there exists a Lebesgue measurable set that is not a Borel set.

*Proof.* Recall that

$$F(x) = \begin{cases} \text{the ternary expansion of } x & \text{if } x \in C_2 \\ F(y) \text{ where } y \text{ is the greatest element of } C_2 \text{ such that } y < x & \text{if } x \notin C_2 \end{cases}$$

We see that $F(x)$ is nondecreasing. Since $x$ is strictly increasing, we have that $G(x) = x + F(x)$ is strictly increasing. We have shown that $F(x)$ is continuous, so $G(x)$ is continuous (since the sum of two continuous functions is continuous). Next, observe that $G(0) = 0$ and $G(1) = 2$. By the Intermediate Value Theorem, we have that $G$ is onto $[0,2]$.

Recall that $C_2^c$ is comprised of open intervals $(a,b)$ where $F(a) = F(b)$. Let $(a,b) \in C_2^c$ have measure $l$. We have

$$\begin{aligned} G(a) &= a + F(a) \\ G(b) &= b + F(b) \end{aligned}$$

Since $F(a) = F(b)$, we see that $G(b) - G(a) = a - b = l$. Furthermore, since $G$ is strictly increasing and continuous, it must be that the open interval $(G(a), G(b))$ has measure $l$. Repeating this argument for all open subsets of $C_2^c$, we see that $m(G(C_2^c)) = m(C_2^c) = 1$. Since $G$ is onto $[0,2]$, we have $m(C_1) = m(G(C_2)) = m([0,2] \setminus G(C_2^c)) = 1$.

Since $G$ is continuous and onto $[0,2]$, $G^{-1}$ is continuous, so we will take $\Phi$ in the problem statement to be $G^{-1}$. Let $N$ denote some non-measurable subset of $C_1$ and consider the function $\chi_{G^{-1}(N)}$. We see that $\chi_{G^{-1}(N)}$ is measurable since

$$\{\chi_{G^{-1}(N)} > a\} = \begin{cases} [0,2] & \text{if } -\infty \leq a \leq 0 \\ \{x : G(x) \in N\} & \text{if } 0 < a \leq 1 \\ \phi & \text{if } a > 1 \end{cases}$$

The first and third cases are obviously measurable. To see that the second case is measurable, observe that $\{x : G(x) \in N\} \subseteq C_2$, which has measure zero. Subsets of sets with measure zero are measurable (also with measure zero), and so it must be that $\{x : G(x) \in N\}$ is measurable. So, we will take $f$ in the problem statement to be $\chi_{G^{-1}(N)}$.

We show next that $\chi_{G^{-1}(N)} \circ G^{-1}$ is non-measurable. Consider the set $\{\chi_{G^{-1}(N)} \circ G^{-1} > 0\}$. Since $\chi_{G^{-1}(N)} \circ G^{-1}$ outputs only 0 or 1, this set is equivalent to $\{\chi_{G^{-1}(N)} \circ G^{-1} = 1\}$. Translating the notation, we see that this set is all of $N$, which is non-measurable by definition. Hence, $\chi_{G^{-1}(N)} \circ G^{-1}$ is non-measurable.

Consider the set $G^{-1}(N)$. Since $G^{-1}(N) \subseteq C_2$ and $C_2$ has measure zero, it must be that $G^{-1}(N)$ is Lebesgue measurable with measure zero. We might proceed by demonstrating that $G^{-1}(N)$ cannot be obtained by from open sets of $\mathbb{R}$ using the properties of $\sigma$-algebras. (I cannot determine how to go about this.) □

Problem 1

Consider the exterior Lebesgue measure $m^*$ introduced in Chapter 1. Prove that a set $E$ in $\mathbb{R}^d$ is Carathéodory measurable if and only if $E$ is Lebesgue measurable in the sense of Chapter 1.

*Proof.* ($\Rightarrow$) Suppose $E$ is Carathéodory measurable. Consider first the case where $E$ has finite measure. We can find open sets $O_n$ so that $m^*(O_n) \leq m(E) + \frac{1}{n}$. Define $G$ to be the $G_\delta$ set $\bigcap_{n=1}^{\infty} O_n$. Since $E \subseteq G \subseteq O_n$ for all $n$, we have that $m^*(E) \leq m^*(G) \leq m^*(O_n) \leq m^(E) + \frac{1}{n}$ for all $n$. Hence, $m^*(G) = m^*(E)$. Now, since $E$ is Carathéodory measurable, we have

$$
\begin{aligned}
m^*(G) &= m^*(E \cap G) + m^*(E^c \cap G) \\
m^*(E) &= m^*(E) + m^*(G - E) \qquad\qquad \text{(since } m^*(E) = m^*(G) \text{ and } E \subseteq G) \\
0 &= m^*(G - E)
\end{aligned}
$$

We see that $E$ differs from a $G_\delta$ set by a set of measure 0. Hence, $E$ is Lebesgue measurable.

Now consider the case where $E$ has infinite measure. Define $E_n$ to be the set $E \cap [-n, n]^d$. For all $n$, we see that $E_n$ has finite Carathéodory measure, and so is Lebesgue measurable by the previous case. Now, $E$ is the countable union of the Lebesgue measurable $E_n$, so $E$ is Lebesgue measurable.

($\Leftarrow$) Suppose $E$ is Lebesgue measurable and let $A \subseteq \mathbb{R}^d$ be given. Using the same method as before, we can construct a $G_\delta$ set $G$ with $A \subseteq G$ and $m^*(A) = m^*(G)$. Now, observe that $G$ is the union of disjoint sets $E \cap G$ and $E^c \cap G$. Since both $E$ and $G$ are Lebesgue measurable, we have

$$m^*(A) = m^*(G) = m^*(E \cap G) + m^*(E^c \cap G) \tag{1}$$

Now, $m^*(A) \leq m^*(E \cap A) + m^*(E^c \cap A)$ by subadditivity. Since $A \subseteq G$, we see that $m^*(E \cap G) \geq m^*(E \cap A)$ and $m^*(E^c \cap G) \geq m^*(E^c \cap A)$. Hence, $m^*(A) = m^*(G) = m^*(E \cap G) + m^*(E^c \cap G) \geq m^*(E \cap A) + m^*(E^c \cap A)$. Therefore, $m^*(A) = m^*(E \cap A) + m^*(E^c \cap A)$, and so $E$ is Carathéodory measurable. ☐

Problem 2 (Tchebychev Inequality)

Suppose $f \geq 0$ and $f$ is integrable. If $\alpha > 0$ and $E_\alpha = \{x \mid f(x) > \alpha\}$, prove that

$$m(E_\alpha) \leq \tfrac{1}{\alpha} \int f.$$

*Proof.* Since $f$ is integrable, it is measurable. This implies, in particular, that $E_\alpha$ is measurable, and so we have that $m(E_\alpha) = \int \chi_{E_\alpha}$. Now, from the definition of $E_\alpha$, we have that $0 \leq \alpha \chi_{E_\alpha} \leq f$, which in turn gives $\alpha \int \chi_{E_\alpha} \leq \int f$. By our previous observation, we can replace $\int \chi_{E_\alpha} dx$ with $m(E_\alpha)$ to get $m(E_\alpha) \leq \frac{1}{\alpha} \int f$, as desired. ☐

Problem 1

Integrability of $f$ on $\mathbb{R}$ does not necessarily imply the convergence of $f(x)$ to 0 as $x \to \infty$.

a. There exists a positive continuous function $f$ on $\mathbb{R}$ so that $f$ is integrable on $\mathbb{R}$, but yet $\overline{\lim}_{x \to \infty} f(x) = \infty$.

*Proof.* Define the function $f$ to take on the value $n$ if $x \in [n, n + \frac{1}{n^3})$ for $n \geq 2$. Elsewhere, the function is zero except for the line segments required to make the function continuous. We define these segments in such a that the graph resembles a sequence of trapezoids with height $n$ and bases of length $\frac{1}{n^3}$ and $\frac{3}{n^3}$.

By construction, $f$ is positive and continuous on $\mathbb{R}$. Now,

$$\begin{aligned}\int_{-\infty}^{\infty} f dx &= \sum_{n=2}^{\infty} n\left(\frac{\frac{1}{n^3}+\frac{3}{n^3}}{2}\right) \\ &= \sum_{n=2}^{\infty} \frac{2}{n^2} \\ &< \infty\end{aligned}$$

Hence, $f$ is integrable. The fact that $\overline{\lim}_{x\to\infty} f(x) = \infty$ is also immediate (for any $N$, there are infinitely-many $x \in (N, \infty)$ such that $f(x) > N$), so the claim is proven.

□

b. However, if we assume that $f$ is uniformly continuous on $\mathbb{R}$ and integrable, then $\overline{\lim}_{|x|\to\infty} f(x) = 0$.

*Proof.* Suppose, for the sake of contradiction, that $\overline{\lim}_{x\to\infty} f(x) = c > 0$ (we consider first only the case where $x \to +\infty$). Choose some $d$ so that $0 < d < c$. Then, there is a sequence $x_1, x_2, \ldots$ with each $x_i$ far apart (to be made precise later) so that $f(x_i) \geq d$ for all $i$. Choose $\epsilon_0$ so that $0 < \epsilon_0 < d$. Since $f$ is uniformly continuous, there is some $\delta_0 > 0$ so that, for each $x_i$, $|f(x_i) - f(y)| < \epsilon_0$ for all $y \in N(x_i, \delta_0)$. Since $f(x_i) \geq d > \epsilon_0$, we have that $f(y) > \epsilon_0$ for all $y \in N(x_i, \delta_0)$. Hence, the area contributed by the function over the interval $N(x_i, \delta_0)$ is at least $2\delta_0\epsilon_0$. Now, if we choose the $x_i$ far enough apart so that each of the $N(x_i, \delta_0)$ are disjoint, we have that

$$\int_0^{\infty} f(x) \geq \sum_{n=1}^{\infty} 2\delta_0\epsilon_0 = \infty$$

which contradicts the fact that $f$ is integrable.

We can force the same contradiction when $x \to -\infty$, and so we conclude that $\overline{\lim}_{|x|\to\infty} f(x)dx = 0$. □

Problem 2

Suppose $f \geq 0$, and let $E_{2^k} = \{x \mid f(x) > 2^k\}$ and $F_k = \{x \mid 2^k < f(x) \leq 2^{k+1}\}$. If $f$ is finite almost everywhere, then

$$\bigcup_{k=-\infty}^{\infty} F_k = \{f(x) > 0\},$$

and the sets $F_k$ are disjoint.

*Proof.* Since $f$ is a function, it has a unique output for each input $x$. Hence, $2^k < f(x) \leq 2^{k+1}$ for a *single* value of $k$. That is, the $F_k$ are disjoint.

($\subseteq$) Let $x \in \bigcup_{k=-\infty}^{\infty} F_k$. Then $2^k < f(x) \leq 2^{k+1}$ for some $k$, so $f(x) \neq 0$. Since $f \geq 0$, we have that $f(x) > 0$. That is, $x \in \{f(x) > 0\}$.

($\supseteq$) Let $x \in \{f(x) > 0\}$. Then $f(x) > 0$, and so $2^k < f(x) \leq 2^{k+1}$ for some $k$. That is, $x \in \bigcup_{k=-\infty}^{\infty} F_k$. □

Prove that $f$ is integrable if and only if

$$\sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty$$

if and only if

$$\sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) < \infty$$

*Proof.* (i $\Rightarrow$ ii) Suppose $f$ is integrable. Let $x \in F_k$ for some $k$. By definition of $F_k$, $2^k < f(x)$. Since the $F_k$ are disjoint, it follows that

$$\begin{aligned}\infty &> \int f dx \\ &> \sum_{k=-\infty}^{\infty} \int 2^k \chi_{F_k} dx \\ &= \sum_{k=-\infty}^{\infty} 2^k \int \chi_{F_k} dx \\ &= \sum_{k=-\infty}^{\infty} 2^k m(F_k)\end{aligned}$$

(ii $\Rightarrow$ i) Suppose $\sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty$. Then $2 \sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty$. Let $x \in F_k$ for some $k$. By definition of $F_k$, $2^{k+1} \geq f(x)$. Since the $F_k$ are disjoint, it follows that

$$\begin{aligned}\int f dx &< \sum_{k=-\infty}^{\infty} \int 2^{k+1} \chi_{F_k} dx \\ &= \sum_{k=-\infty}^{\infty} 2^{k+1} \int \chi_{F_k} dx \\ &= \sum_{k=-\infty}^{\infty} 2^{k+1} m(F_k) \\ &< \infty\end{aligned}$$

(ii $\Leftrightarrow$ iii) Suppose $\sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty$. Observe that $E_{2^k} = \bigcup_{n \geq k} F_n$. Since the $F_n$ are disjoint and measurable, it follows that

$$\begin{aligned}m(E_{2^k}) &= m(\bigcup_{n \geq k} F_n) \\ &= \sum_{n \geq k} m(F_n)\end{aligned}$$

So

$$\begin{aligned}\sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) &= \sum_{k=-\infty}^{\infty} \sum_{n \geq k} 2^k m(F_n) \\ &= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{n} 2^k m(F_n) \\ &= \sum_{n=-\infty}^{\infty} m(F_n) \sum_{k=-\infty}^{n} 2^k \\ &= \sum_{n=-\infty}^{\infty} 2^{n+1} m(F_n) \\ &= 2 \sum_{n=-\infty}^{\infty} 2^n m(F_n)\end{aligned}$$

Hence, if either of $\sum_{k=-\infty}^{\infty} 2^k m(E_{2^k})$ or $\sum_{k=-\infty}^{\infty} 2^k m(F_k)$ is finite, then the other is also finite. $\square$

Use this result to verify the following assertions. Let

$$f(x) = \begin{cases} |x|^{-a} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$g(x) = \begin{cases} |x|^{-b} & \text{if } |x| > 1 \\ 0 & \text{otherwise} \end{cases}$$

Then $f$ is integrable on $\mathbb{R}^d$ if and only if $a < d$; also $g$ is integrable on $\mathbb{R}^d$ if and only if $b > d$.

*Proof.* Let $x \in F_k$. Then

$$\begin{aligned} 2^k &< |x|^{-a} &&\leq 2^{k+1} \\ 2^{\frac{-k}{a}} &> |x| &&\geq 2^{\frac{-k-1}{a}} \end{aligned}$$

Hence

$$\begin{aligned} m(B(0; 2^{\frac{-k}{a}})) &> m(F_k) &&\geq m(B(0; 2^{\frac{-k-1}{a}})) \\ v_d 2^{\frac{-dk}{a}} &> m(F_k) &&\geq v_d 2^{\frac{-d(k+1)}{a}} \end{aligned}$$

where $v_d$ is the volume of the unit ball. Now

$$\sum_{k=-\infty}^{\infty} 2^k m(F_k) = \sum_{k=-\infty}^{0} 2^k m(F_k) + \sum_{k=1}^{\infty} 2^k m(F_k)$$

Since $|x| \leq 1$, $m(F_k) \leq v_d$ for all $k$. Hence

$$\begin{aligned} \sum_{k=-\infty}^{\infty} 2^k m(F_k) &\leq v_d \sum_{k=-\infty}^{0} 2^k + \sum_{k=1}^{\infty} 2^k m(F_k) \\ &= 2v_d + \sum_{k=1}^{\infty} 2^k m(F_k) \end{aligned}$$

So it suffices to show that $\sum_{k=1}^{\infty} 2^k m(F_k)$ converges.

$$\begin{aligned} \sum_{k=1}^{\infty} 2^k (v_d 2^{\frac{-d(k+1)}{a}}) &\leq \sum_{k=1}^{\infty} 2^k m(F_k) &&< \sum_{k=1}^{\infty} 2^k (v_d 2^{\frac{-dk}{a}}) \\ v_d 2^{\frac{-d}{a}} \sum_{k=1}^{\infty} 2^{k(1-\frac{d}{a})} &\leq \sum_{k=1}^{\infty} 2^k m(F_k) &&< v_d \sum_{k=1}^{\infty} 2^{k(1-\frac{d}{a})} \end{aligned}$$

We see that the upper and lower bounds converge if and only if $0 < a < d$, forcing the convergence of $\sum_{k=-\infty}^{\infty} 2^k m(F_k)$, which in turn implies that $f$ is integrable.

Let $x \in E_{2^k}$. Then

$$2^k < |x|^{-b}$$
$$2^{\frac{-k}{b}} > |x|$$

Observe also that $E_{2^k}$ is empty for $k \leq 0$. Hence, $E_{2^k} = B(0; 2^{\frac{-k}{b}}) \setminus B(0;1)$, and so $m(E_{2^k}) = v_d 2^{\frac{-dk}{b}} - v_d$. Now

$$\begin{aligned}\sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) &= \sum_{k=-\infty}^{-1} 2^k m(E_{2^k}) \\ &= \sum_{k=-\infty}^{-1} 2^k (v_d 2^{\frac{-dk}{b}} - v_d) \\ &= v_d \sum_{k=-\infty}^{-1} 2^{k(1-\frac{d}{b})} - v_d \sum_{k=-\infty}^{-1} 2^k\end{aligned}$$

which converges if and only if $b > d$, which in turn implies that $f$ is integrable.

□

Problem 3

a. Prove that if $f$ is integrable on $\mathbb{R}^d$, real-valued, and $\int_E f(x)dx \geq 0$ for every measurable $E$, then $f(x) \geq 0$ a.e. $x$.

*Proof.* Define $F$ to be the set $\{x \mid f(x) < 0\}$. Since $f$ is integrable, $f$ is measurable, and so $F$ is measurable. We claim that $m(F) = 0$.

Since $F$ is measurable, we have that $\int_F f(x)dx \geq 0$ by hypothesis. Now, observe that, for any $n \geq 1$, $nf\chi_F \leq f$. It follows that

$$\begin{aligned}\int nf\chi_F dx &\leq \int f dx \\ n\int_F f dx &\leq \int f dx \\ \int_F f dx &\leq \frac{1}{n}\int f dx \\ \int_F f dx &\leq 0\end{aligned}$$

Hence, $\int_F f dx = 0$. Since $f(x) < 0$ for all $x \in F$, we conclude that $m(F) = 0$. That is, $f(x) \geq 0$ almost everywhere.

□

b. As a result, if $\int_E f(x)dx = 0$ for every measurable $E$, then $f(x) = 0$ a.e.

*Proof.* From the first part, we see that $f(x) \geq 0$ almost everywhere. Let $G$ be the set $\{x \mid f(x) > 0\}$. It suffices to show that $m(G) = 0$.

As before, $f$ is measurable, so $G$ is measurable. By hypothesis, we have that $\int_G f dx = 0$. Since $f(x) > 0$ for all $x \in G$, we conclude that $m(G) = 0$. Hence, the set of $x$ so that $f(x) \neq 0$ has measure 0. That is, $f(x) = 0$ almost everywhere. □

Problem 4

a. Let $a_n, b_n \in \mathbb{R}$ such that $a_n \to a \in \mathbb{R}$. Prove that

$$\underline{\lim}(a_n + b_n) = a + \underline{\lim} b_n$$

*Proof.* Since $a_n \to a$, we have $a = \lim_{n\to\infty} a_n = \overline{\lim}_{n\to\infty} a_n = \underline{\lim}_{n\to\infty} a_n$. Since $\overline{\lim}(a_n) = -\underline{\lim}(-a_n)$, we have

$$\begin{aligned}\underline{\lim}(a_n + b_n) &= -\overline{\lim}(-a_n - b_n)\\ &\geq -\overline{\lim}(-a_n) - \overline{\lim}(-b_n)\\ &= \underline{\lim} a_n + \underline{\lim} b_n\\ &= a + \underline{\lim} b_n\end{aligned}$$

Now, construct a subsequence $(b_{n_k})$ of $(b_n)$ with $\lim_{k\to\infty} b_{n_k} = \underline{\lim}_{n\to\infty} b_n$. Let $(a_{n_k})$ be the subsequence induced by the indices chosen for $(b_{n_k})$.

$$\begin{aligned}a + \underline{\lim} b_n &= \underline{\lim} a_n + \underline{\lim} b_n\\ &= \lim a_{n_k} + \lim b_{n_k}\\ &= \lim(a_{n_k} + b_{n_k})\\ &\geq \underline{\lim}(a_n + b_n)\end{aligned}$$

Therefore, $\underline{\lim}(a_n + b_n) = a + \underline{\lim} b_n$. □

b. Let $f, f_n$ be integrable functions. Assume $f_n(x) arrow f(x)$ a.e. and $\int |f_n| dx \to \int |f| dx$. Prove that $\int |f_n - f| dx \to 0$.

*Proof.* Define the function $g_n$ to be $|f| + |f_n| - |f - f_n|$. Then $g_n \to 2|f|$ as $n \to \infty$. By Fatou's lemma

$$\int g dx \leq \underline{\lim} \int g_n dx$$

Hence

$$\begin{aligned}2\int |f|\, dx &\leq \underline{\lim} \int (|f| + |f_n| - |f - f_n|)\, dx\\ &= \underline{\lim}(\int |f|\, dx + \int |f_n|\, dx - \int |f - f_n|\, dx)\\ &= \int |f|\, dx + \underline{\lim}(\int |f_n|\, dx - \int |f - f_n|\, dx)\\ &= \int |f|\, dx + \int |f|\, dx + \underline{\lim}(-\int |f - f_n|\, dx) \qquad \text{(by part (a))}\end{aligned}$$

Now,

$$\begin{aligned}0 &\leq \underline{\lim}(-\int |f - f_n|\, dx)\\ 0 &\geq \overline{\lim} \int |f - f_n|\, dx\\ &\geq \lim_{n\to\infty} \int |f - f_n| dx\end{aligned}$$

Therefore, as $n \to \infty$, $\int |f - f_n|\, dx \to 0$. □

<u>Problem 1</u>

a. For $m(E) < \infty$, show that

$$L^\infty(E) \subset L^r(E) \subset L^p(E) \subset L^1(E)$$

where $1 < p < r < \infty$. Show, for $E = (0, 1]$, by example that all the inclusions can be strict.

**Claim.** $L^\infty(E) \subset L^r(E)$

*Proof.* Let $f \in L^\infty(E)$. We have that $f$ is measurable and, for some $M$, $|f(x)| \leq M$ almost everywhere on $E$. Hence

$$\begin{aligned} |f(x)|^r &\leq M^r \\ \int_E |f(x)|^r dx &\leq M^r m(E) \\ &< \infty \end{aligned}$$

Hence, $f \in L^r(E)$.

To see that this inclusion is strict, consider the function $f = \frac{1}{x^{\frac{1}{2r}}}$ on $E = (0, 1]$. We see that $f$ is unbounded, so $f \notin L^\infty(E)$, but

$$\begin{aligned} \int_E |f(x)|^r \, dx &= \int_E \left| \frac{1}{x^{\frac{1}{2r}}} \right|^r dx \\ &= \int_E \left| \frac{1}{x^{\frac{1}{2}}} \right| dx \\ &= 2 \\ &< \infty \end{aligned}$$

so $f \in L^r$. □

**Claim.** $L^r(E) \subset L^p(E)$

*Proof.* Let $f \in L^r(E)$. We have that $f$ is measurable and $\left(\int_E |f(x)|^r dx\right)^{\frac{1}{r}} < \infty$. Now, let $n = \frac{r}{p}$. Observe that $\frac{1}{n} = \frac{p}{r} < 1$, so there exists a number $q$ so that $\frac{1}{n} + \frac{1}{q} = 1$. Let $g(x) = (f(x))^p$ for all $x$. We show that $g \in L^n(E)$. Since $f \in L^r(E)$, $f$ is measurable, and so $g = f^p$ is measurable. Furthermore,

$$\begin{aligned} \int_E |g|^n dx &= \int_E |(f(x))^p|^{\frac{r}{p}} \, dx \\ &= \int_E |f(x)|^r dx \\ &< \infty \end{aligned}$$

We also see that the constant function 1 is in $L^q(E)$, since 1 is measurable and

$$\begin{aligned} \int_E |1|^q dx &= \int_E dx \\ &= m(E) \\ &< \infty \end{aligned}$$

Next, apply Hölder's Inequality to $g \cdot 1$ to obtain

$$\begin{aligned} \int_E |g(x) \cdot 1| dx &\leq ||g||_n \, ||1||_q \\ &= \left( \int_E |g(x)|^n \, dx \right)^{\frac{1}{n}} \left( \int_E |1|^q \, dx \right)^{\frac{1}{q}} \\ &< \infty \end{aligned}$$

Since $\int_E |g(x) \cdot 1| dx = \int_E |f(x)|^p dx$, we have that $f \in L^p$.

To see that this inclusion is strict, consider the function $f = \frac{1}{x^{\frac{1}{r}}}$ on $E = (0, 1]$. Observe that $f$ is measurable. Now, we see that $f \notin L^r(E)$, since

$$\begin{aligned}\int_E |f|^r \, dx &= \int_E \left|\frac{1}{x^{\frac{1}{r}}}\right|^r dx \\ &= \int_E \left|\frac{1}{x}\right|^r dx \\ &= \infty\end{aligned}$$

Now,

$$\begin{aligned}\int_E |f|^p \, dx &= \int_E \left|\frac{1}{x^{\frac{1}{r}}}\right|^p dx \\ &= \int_E \left|\frac{1}{x^{\frac{p}{r}}}\right| dx \\ &< \infty\end{aligned}$$

since $\frac{p}{r} < 1$. Hence, $f \in L^p(E)$. □

**Claim.** $L^p \subset L^1$

*Proof.* As in the previous proof, but replace every occurrence of "$r$" with "$p$" and every occurrence of "$p$" with "1". Similarly, this inclusion is strict. □

b. Show that in general (i.e., if $m(E) = \infty$)

$$L^\infty \cap L^1 \subset L^p \subset L^\infty + L^1 = \{f : f = g + h, g \in L^\infty, h \in L^1\}$$

**Claim.** $L^\infty \cap L^1 \subset L^p$

*Proof.* Let $f \in L^\infty \cap L^1$. We know that $|f|$ is bounded (say by $M$) and integrable. It follows that

$$\begin{aligned}\int_E |f|^p dx &= \int_E |f||f|^{p-1} dx \\ &\leq M^{p-1} \int_E |f| dx \\ &< \infty\end{aligned}$$

Therefore, $f \in L^p$. □

**Claim.** $L^p \subset L^\infty + L^1$

*Proof.* Let $f \in L^p$. We know that $\int_E |f|^p dx < \infty$. Observe that

$$E = \{x : |f(x)|^p < 1\} \cup \{x : |f(x)|^p \geq 1\}$$

Define the function $g$ to be $f$ restricted to $\{x : |f(x)|^p < 1\}$ and the function $h$ to be $f$ restricted to $\{x : |f(x)|^p \geq 1\}$. First, observe that that $f = g + h$ with $g$ and $h$ both measurable. Next, we see that $g \in L^\infty$, since $|g(x)| < 1$ for all $x$ in its domain. Finally, if we denote $\{x : |f(x)|^p \geq 1\}$ by $H$,

$$\begin{aligned}\int_H |h(x)| dx &= \int_H |f(x)| dx \\ &\leq \int_H |f(x)|^p dx \\ &\leq \int_E |f(x)|^p dx \\ &< \infty\end{aligned}$$

Hence, $h \in L^1$. □

Problem 2

Let $f \in L^2([0,1])$. Prove that

$$\left(\int_{[0,1]} xf(x)dx\right)^2 \leq \frac{1}{3}\int_{[0,1]} |f(x)|^2 dx$$

*Proof.* Observe first that the function $x$ is in $L^2([0,1])$, since $x$ is measurable and

$$\begin{aligned} \int_{[0,1]} |x|^2 \, dx &= \int_{[0,1]} x^2 dx \\ &= \frac{1}{3} \\ &< \infty \end{aligned}$$

□

Now,

$$\begin{aligned} \left(\int_{[0,1]} xf(x)dx\right)^2 &= \left|\int_{[0,1]} xf(x)dx\right|^2 \\ &\leq \left(\int_{[0,1]} |xf(x)|dx\right)^2 \\ &\leq (||x||_2 \, ||f(x)||_2)^2 \qquad \text{(by Hölder's Inequality)} \\ &= \left(\left(\int_E |x|^2 \, dx\right)^{\frac{1}{2}} \left(\int_E |f(x)|^2 \, dx\right)^{\frac{1}{2}}\right)^2 \\ &= \int_{[0,1]} |x|^2 dx \int_{[0,1]} |f(x)|^2 dx \\ &= \frac{1}{3}\int_{[0,1]} |f(x)|^2 dx \end{aligned}$$

Problem 3

Let $E$ be a measurable set of finite measure and let $1 < p < \infty$. Assume $f_n \in L^p(E)$ such that $||f_n||_p \leq 1$ and $f_n(x) \to 0$ almost everywhere. Prove that $||f_n||_1 \to 0$.

*Proof.* By Egorov's Theorem, we can find, for any $\epsilon_0 > 0$, a subset $A_{\epsilon_0}$ of $E$ with $m(E \setminus A_{\epsilon_0}) < \epsilon_0$ so that $f_n \to 0$ uniformly on $A_{\epsilon_0}$. This implies that, for any $\epsilon > 0$, there is an $N$ so that for all $n \geq N$,

$$\begin{aligned}\int_{A_{\epsilon_0}} |f_n| dx &\leq \int_{A_{\epsilon_0}} \epsilon dx \\ &= \epsilon m(A_{\epsilon_0}) \\ &< \epsilon m(E)\end{aligned}$$

Hence, $\int_{A_{\epsilon_0}} |f_n| dx \to 0$.

For the remainder of the domain, observe that

$$\int_{E \setminus A_{\epsilon_0}} |f_n| dx = \int_E |f_n \chi_{E \setminus A_{\epsilon_0}}| dx$$

Since $1 < p < \infty$, we can choose $q$ so that $\frac{1}{p} + \frac{1}{q} = 1$. We already know that $f_n \in L^p(E)$ for all $n$. In order to apply Hölder's Inequality, we need that $\chi_{E \setminus A_{\epsilon_0}} \in L^q(E)$. This is true since

$$\begin{aligned}\int_E |\chi_{E \setminus A_{\epsilon_0}}|^q dx &< \int_E |1|^q dx \\ &= \int_E dx \\ &= m(E) \\ &< \infty\end{aligned}$$

Applying Holder's Inequality, we see that

$$\begin{aligned}\int_{E \setminus A_{\epsilon_0}} |f_n| dx &= \int_E |f_n \chi_{E \setminus A_{\epsilon_0}}| dx \\ &\leq ||f_n||_p \left|\left|\chi_{E \setminus A_{\epsilon_0}}\right|\right|_q \\ &\leq \left|\left|\chi_{E \setminus A_{\epsilon_0}}\right|\right|_q \\ &= \left(\int_E \left|\chi_{E \setminus A_{\epsilon_0}}\right|^q dx\right)^{\frac{1}{q}} \\ &= \left(\int_E \chi_{E \setminus A_{\epsilon_0}} dx\right)^{\frac{1}{q}} \\ &= (m(E \setminus A_{\epsilon_0}))^{\frac{1}{q}}\end{aligned}$$

Since $m(E \setminus A_{\epsilon_0})$ can be made arbitrarily small, we conclude that $\int_{E \setminus A_{\epsilon_0}} |f_n| dx \to 0$. Taken together with the fact that $\int_{A_{\epsilon_0}} |f_n| dx \to 0$, we have that $||f_n||_1 \to 0$. □

Problem 4

Let $f_n \to f$ in $L^p$, $1 \leq p < \infty$, and let $\{g_n\}$ be a sequence of measurable functions such that $|g_n| \leq M$ for all $n$ with $g_n \to g$ almost everywhere.

a. Prove $||(g_n - g)f||_p \to 0$.

*Proof.* Observe first that

$$\begin{aligned} ||(g_n - g)f||_p &= \left( \int_E |(g_n - g)f|^p \, dx \right)^{\frac{1}{p}} \\ &= \left( \int |g_n - g|^p |f|^p dx \right)^{\frac{1}{p}} \end{aligned}$$

The proof proceeds by establishing the hypotheses for the Dominated Convergence Theorem. Define $F_n = |g_n - g|^p |f|^p$. Now, $f \in L^p$ implies that $f$ is finite almost everywhere. Taken with the fact that $g_n \to g$ almost everywhere, we have that $F_n \to 0$ almost everywhere. Next, observe that, since $|g_n| \leq M$ and $g_n \to g$ almost everywhere, $|g_n - g| \leq 2M$ almost everywhere. Define the function $G = (2M)^p |f|^p$. We see that $|F_n| \leq G$ almost everywhere and $G$ is integrable (since $f \in L^p$). It follows that

$$\begin{aligned} \lim_{n\to\infty} \int |g_n - g|^p |f|^p dx &= \lim_{n\to\infty} \int F_n dx \\ &= \int \lim_{n\to\infty} F_n dx \qquad \text{(by the Dominated Convergence Theorem)} \\ &= 0 \end{aligned}$$

Therefore, $||(g_n - g)f||_p \to 0$. □

b. Prove $g_n f_n \to fg$ in $L^p$.

*Proof.*

$$\begin{aligned} ||g_n f_n - fg||_p &= ||g_n f_n - fg + g_n f - g_n f||_p \\ &= ||(g_n - g)f + (f_n - f)g_n||_p \\ &\leq ||(g_n - g)f||_p + ||(f_n - f)g_n||_p \\ &\leq ||(g_n - g)f||_p + ||(f_n - f)M||_p \\ &= ||(g_n - g)f||_p + M\, ||f_n - f||_p \end{aligned}$$

The first term goes to 0 by part (a) and the second term goes to 0 by the assumption that $f_n \to f$ in $L^p$. Hence, $g_n f_n \to fg$ in $L^p$. □

Problem 1

Consider the function defined over $\mathbb{R}$ by

$$f(x) = \begin{cases} x^{-1/2} & \text{if } 0 < x < 1, \\ 0 & \text{otherwise} \end{cases}$$

For a fixed enumeration $\{r_n\}_{n=1}^{\infty}$ of the rationals $\mathbb{Q}$, let

$$F(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n).$$

Prove that $F$ is integrable, hence the series defining $F$ converges for almost every $x \in \mathbb{R}$. However, observe that this series is unbounded on every interval, and in fact, any function $\tilde{F}$ that agrees with $F$ almost everywhere is unbounded in any interval.

*Proof.* Define $s_k = \sum_{n=1}^{k} 2^{-n} f(x - r_n)$. Since $2^{-n}$ and $f(x - r_n)$ are both measurable for each $n$, we have that $s_k$ is measurable for each $k$. Furthermore, $s_k \geq 0$ for each $k$. It follows that

$$\begin{aligned}
\int_{\mathbb{R}} F(x)dx &= \int_{\mathbb{R}} \sum_{n=1}^{\infty} 2^{-n} f(x - r_n)dx \\
&= \sum_{n=1}^{\infty} \int_{\mathbb{R}} 2^{-n} f(x - r_n)dx \qquad \text{(by the Monotone Convergence Theorem)} \\
&= \sum_{n=1}^{\infty} 2^{-n} \int_{\mathbb{R}} f(x)dx \\
&= \sum_{n=1}^{\infty} 2^{-n} \cdot 2 \\
&= 2
\end{aligned}$$

Hence, $F$ is integrable. It follows directly that $F$ converges for almost every $x \in \mathbb{R}$ (if there was a set of positive measure on which $F$ did not converge, then the integral of $F$ over that set would be infinite).

Let $\tilde{F}$ be as described. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, any interval of $\mathbb{R}$ contains a rational number $r_N$ a real number $x_N$ such that, for any $\epsilon > 0$, $|x_N - r_N| < \epsilon$. Hence, for any $x \in B(r_N, \epsilon)$,

$$\begin{aligned}
\tilde{F}(x) &= \sum_{n=1}^{\infty} 2^{-n} f(x - r_n)dx \\
&> 2^{-N} f(x - r_N) \\
&= 2^{-N} \epsilon^{\frac{-1}{2}}
\end{aligned}$$

which is unbounded as $\epsilon \to 0$. □

Problem 2

a. Let $f \in L^r \cap L^\infty$ for some $r < \infty$. Prove that

$$||f||_p \leq ||f||_r^{\frac{r}{p}} \, ||f||_\infty^{1-\frac{r}{p}}$$

for all $r < p < \infty$.

*Proof.* Since $f \in L^\infty$, $|f| \leq M$ for some $M$. Now,

$$\begin{aligned}
\int_E |f|^p \, dx &= \int |f|^{p-r} |f|^r dx \\
&\leq \int M^{p-r} |f|^r dx \\
&= M^{p-r} \int |f|^r dx
\end{aligned}$$

which is finite, since $f \in L^r$. Hence,

$$\begin{aligned}
||f||_p &= \left( \int_E |f|^p \, dx \right)^{\frac{1}{p}} \\
&\leq (M^{p-r} \int |f|^r dx)^{\frac{1}{p}} \\
&= M^{1-\frac{r}{p}} (\int |f|^r dx)^{\frac{1}{p}} \\
&= ||f||_\infty^{1-\frac{r}{p}} \, ||f||_r^{\frac{r}{p}}
\end{aligned}$$

□

b. Assume $f \in L^r \cap L^\infty$ for some $r < \infty$. Prove

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$$

*Proof.* Observe from part (a) that

$$\begin{aligned} ||f||_p &\leq ||f||_r^{\frac{r}{p}} \, ||f||_\infty^{1-\frac{r}{p}} \\ \overline{\lim}\, ||f||_p &\leq \overline{\lim}\, ||f||_r^{\frac{r}{p}} \, ||f||_\infty^{1-\frac{r}{p}} \\ &\leq ||f||_\infty \end{aligned}$$

Now, for $0 < t < ||f||_\infty$, define the set $A = \{x : |f(x)| \geq t\}$. Observe that, for all $t$, $A$ has positive measure. Suppose this is not the case. We can find $t_0 < ||f||_\infty$ with $|f(x)| < t_0$ almost everywhere, which is a contradiction with the definition of $||f||_\infty$. Furthermore, we see that, for all $t$, $A$ has finite measure. Suppose this is not the case. We see that

$$\begin{aligned} \int |f|^r dx &\geq \int_A |f|^r dx \\ &\geq \int_A t^r dx \\ &= t^r m(A) \\ &= \infty \end{aligned}$$

which is a contradiction with the fact that $f \in L^r$. Now, observe that $|f(x)| \geq t\chi_A(x)$ for all $x$, which implies that $||f||_p \geq ||t\chi_A(x)||_p$. It follows that, for any $t$

$$\begin{aligned} \underline{\lim}\, ||f||_p &\geq \underline{\lim}\, ||t\chi_A||_p \\ &= \underline{\lim} \left( \int_A |t\chi_A(x)|^p dx \right)^{\frac{1}{p}} \\ &= \underline{\lim}\, (t^p m(A))^{\frac{1}{p}} \\ &= \underline{\lim}\, t m(A)^{\frac{1}{p}} \\ &= t \end{aligned}$$

Since $t$ can be chosen arbitrarily close to $||f||_\infty$, it follows that $\underline{\lim}\, ||f||_p \geq ||f||_\infty$. Combining this with the above, we conclude that $\lim_{p\to\infty} ||f||_p = ||f||_\infty$. ☐

Problem 3

Let $f_n, f \in L^p$ with $1 \leq p < \infty$. Assume $f_n(x) \to f(x)$ a.e. Prove $f_n \to f$ in $L^p$ if and only if $\|f_n\|_p \to \|f\|_p$.

*Proof.* Define the function $g_n = |f|^p + |f_n|^p - |f - f_n|^p$. We have, for each $n$, $g_n$ is measurable and $g_n(x) \to 2|f(x)|^p$, so $g_n \geq 0$ for all sufficiently large $n$. By Fatou's Lemma

$$\begin{aligned} \int 2|f|^p dx &\leq \underline{\lim} \int g_n dx \\ &= \underline{\lim} \int |f|^p + |f_n|^p - |f - f_n|^p dx \\ &= \underline{\lim} (\int |f|^p dx + \int |f_n|^p dx - \int |f - f_n| dx) \\ &= 2 \int |f|^p dx + \underline{\lim} - \int |f - f_n|^p dx \qquad \text{(by a previous homework)} \\ &= 2 \int |f|^p dx - \overline{\lim} \int |f - f_n|^p dx \end{aligned}$$

Hence

$$\begin{aligned}\int 2|f|^p dx &\le \int 2|f|^p - \overline{\lim} \int |f-f_n|^p dx \\ \overline{\lim} \int |f-f_n|^p dx &\le 0\end{aligned}$$

and so $\lim \int |f-f_n|^p dx = 0$. Therefore, $||f||_p \to ||f_n||_p$. □

Problem 4

Let $f \in L^1$. Denote by $f_h$ the function $f_h(x) = f(x-h)$. Prove that $\|f - f_h\|_1 \to 0$ as $h \to 0$.

*Proof.* Let $F$ be a continuous function with compact support $E$. We see immediately that $m(E) < \infty$. Let $\epsilon > 0$ be given. Since $F$ is continuous, we can find $\delta > 0$ so that for $|x-y| < \delta$, $|F(x) - F(y)| < \epsilon$. Now, let $0 < h < \delta$. Since $|x - (x-h)| = h < \delta$, we conclude that $|F(x) - F(x-h)| < \epsilon$. Hence,

$$\begin{aligned}||F - F_h||_1 &= \int |F - F_h| dx \\ &= \int_E |F - F_h| dx \\ &\le \int_E \epsilon dx \\ &= \epsilon m(E)\end{aligned}$$

Since $\epsilon$ is arbitrary and $m(E) < \infty$, we conclude that $||F - F_h||_1 \to 0$.

Since continuous functions with compact support are dense in $L^1$, for any $\epsilon > 0$, we can find continuous $F$ with compact support such that $||F - f||_1 < \frac{\epsilon}{3}$. It follows that

$$\begin{aligned}||f - f_h||_1 &= ||f - F + F - F_h + F_h - f_h||_1 \\ &\le ||f - F||_1 + ||F - F_h||_1 + ||F_h - f_h||_1 \\ &= ||f - F||_1 + ||F - F_h||_1 + ||(F - f)_h||_1 \\ &= ||f - F||_1 + ||F - F_h||_1 + ||F - f||_1 \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon\end{aligned}$$

noting that the second term can be bounded in this way because $F$ is continuous with compact support. □

Problem 1

Suppose $f$ is defined on $\mathbb{R}^2$ as follows

$$f(x,y) = \begin{cases} a_n & \text{if } n \ge 0, n \le x < n+1, n \le y < n+1 \\ -a_n & \text{if } n \ge 0, n \le x < n+1, n+1 \le y < n+2 \\ 0 & \text{otherwise} \end{cases}$$

Here $a_n = \sum_{k \le n} b_k$ with $\{b_k\}$ a positive sequence such that $\sum_{k=0}^{\infty} b_k = s < \infty$.

a. Verify that each slice $f^y$ and $f_x$ is integrable. Also, for all $x$, $\int_{\mathbb{R}} f_x(y) dy = 0$, and hence $\int \left( \int_{\mathbb{R}} f(x,y) dy \right) dx = 0$.

*Proof.* Observe that for $0 \le y < 1$

$$f^y(x) = \begin{cases} a_0 & \text{if } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

and for $n \leq y < n+1$ with $n \geq 1$

$$f^y(x) = \begin{cases} a_n & \text{if } n \leq x < n+1 \\ -a_{n-1} & \text{if } n-1 \leq x < n \\ 0 & \text{otherwise} \end{cases}$$

Also, for all $n \geq 0$

$$f_x(y) = \begin{cases} a_n & \text{if } n \leq y < n+1 \\ -a_n & \text{if } n+1 \leq y < n+2 \\ 0 & \text{otherwise} \end{cases}$$

all of which are step functions, and hence measurable. We will compute the values of their integrals as they are needed. These values will turn out to be finite, and so the slices will be shown integrable.

Let $n \leq x < n+1$. We have

$$\begin{aligned} \int_{\mathbb{R}} f_x(y)dy &= \int_{[n,n+1]} a_n dy + \int_{[n+1,n+2]} -a_n dy \\ &= a_n - a_n \\ &= 0 \end{aligned}$$

and so $f_x$ is integrable and

$$\begin{aligned} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(x,y)dy \right) dx &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f_x(y)dy \right) dx \\ &= \int_{\mathbb{R}} 0dx \\ &= 0 \end{aligned}$$

□

b. However,

$$\textstyle\int_{\mathbb{R}} f^y(x)dy = \begin{cases} a_0 & \text{if } 0 \leq y < 1 \\ a_n - a_{n-1} & \text{if } n \geq 1, n \leq y < n+1 \end{cases}$$

Hence, $y \mapsto \int_{\mathbb{R}} f^y(x)dx$ is integrable on $(0, \infty)$ and

$$\textstyle\int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(x,y)dx \right) dy = s$$

*Proof.* Let $0 \leq y < 1$. From our definition of $f^y(x)$ in part (a), we have

$$\begin{aligned} \int_{\mathbb{R}} f^y(x)dx &= \int_{[0,1]} a_0 dx \\ &= a_0 \end{aligned}$$

Similarly, if $n \leq y < n+1$ with $n \geq 1$, we have

$$\begin{aligned} \int_{\mathbb{R}} f^y(x)dx &= \int_{[n-1,n]} -a_{n-1}dx + \int_{[n,n+1]} a_n dx \\ &= a_n - a_{n-1} \end{aligned}$$

Hence, for every y, $f^y$ is integrable. Now, Tonelli's Theorem gives that $y \mapsto \int_{\mathbb{R}} f^y(x)dx$ is measurable. Furthermore,

$$\begin{aligned}
\int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(x,y)dx \right) dy &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f^y(x)dx \right) dy \\
&= \sum_{n=0}^{\infty} \left( \int_{[n,n+1]} \left( \int_{\mathbb{R}} f^y(x)dx \right) dy \right) \\
&= \sum_{n=0}^{\infty} \left( \int_{[n,n+1]} (a_n - a_{n-1})\, dy \right) \\
&= \sum_{n=0}^{\infty} (a_n - a_{n-1}) \\
&= s
\end{aligned}$$

where we define the term $a_{-1}$ to be 0. □

c. Note that $\int_{\mathbb{R}\times\mathbb{R}} |f(x,y)|dxdy = \infty$.

*Proof.*

$$\begin{aligned}
\int_{\mathbb{R}\times\mathbb{R}} |f(x,y)|dxdy &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(x,y)|dx \right) dy \\
&= \int_{\mathbb{R}} \left( \int_{(0,\infty)} |f(x,y)|dx \right) dy \\
&\geq \int_{\mathbb{R}} \left( \int_{(0,\infty)} a_0 dx \right) dy && \text{(since } \{a_n\} \text{ is a monotone increasing sequence)} \\
&= \infty
\end{aligned}$$

□

Problem 2

Suppose $f$ is integrable on $\mathbb{R}^d$. For each $\alpha > 0$, let $E_\alpha = \{x : |f(x)| > \alpha\}$. Prove that

$$\int_{\mathbb{R}^d} |f(x)|dx = \int_0^\infty m(E_\alpha)d\alpha$$

*Proof.* Observe first that, for fixed $\alpha > 0$,

$$\begin{aligned}
E_\alpha &= \{x : |f(x)| > \alpha\} \\
&= \{x : \alpha > f(x) > -\alpha\}
\end{aligned}$$

Since $f$ is integrable, it is measurable, and so $E_\alpha$ is measurable. Now,

$$\begin{aligned}
\int_0^\infty m(E_\alpha)d\alpha &= \int_0^\infty \left( \int_{\mathbb{R}^d} \chi_{E_\alpha}(t)dt \right) d\alpha \\
&= \int_{\mathbb{R}^d} \left( \int_0^\infty \chi_{E_\alpha}(t)d\alpha \right) dt && \text{(by Tonelli's Theorem)} \\
&= \int_{\mathbb{R}^d} \left( \int_0^{|f(t)|} 1 d\alpha \right) dt && \text{(since } \chi_{E_\alpha} = 0 \text{ when } \alpha \geq |f(t)| \text{ for fixed } t) \\
&= \int_{\mathbb{R}^d} |f(t)|dt
\end{aligned}$$

□

Problem 3

Consider the convolution

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\, dy.$$

a. Show that $f * g$ is uniformly continuous when $f$ is integrable and $g$ bounded.

*Proof.* Let $g(x) \leq M$ for all $x$. We have

$$\left| \int_{\mathbb{R}}^d f(x-y)g(y)\, dy \right| \leq M \int_{\mathbb{R}}^d |f(x-y)|\, dy < \infty$$

Let $\epsilon > 0$. We know $\|f_h - f\|_1 \to 0$ as $h \to 0$ (from a previous homework). Hence, we can find $\delta > 0$ such that $\|f_h - f\|_1 < \frac{\epsilon}{M}$ whenever $h < \delta$. Now

$$\begin{aligned} |(f*g)(x_1) - (f*g)(x_2)| &= \left| \int_{\mathbb{R}}^d f(x_1-y)g(y)dy - \int_{\mathbb{R}}^d f(x_2-y)g(y)dy \right| \\ &= \left| \int_{\mathbb{R}}^d (f(x_1-y) - f(x_2-y))g(y)dy \right| \end{aligned}$$

Perform the change of variable $u = x_2 - y$ to get

$$\begin{aligned} \left| \int_{\mathbb{R}}^d (f(u-(x_1-x_2)) - f(u))g(-u-x_2)du \right| &\leq \int_{\mathbb{R}}^d |f(u-(x_1-x_2)) - f(u)||g(-u-x_2)|du \\ &\leq M \int_{\mathbb{R}}^d |f(u-(x_1-x_2)) - f(u)|du \\ &= M\|f_{x_1-x_2} - f\|_1 \end{aligned}$$

Now, $\|f_{x_1-x_2} - f\|_1 < \frac{\epsilon}{M}$ whenever $|x_1 - x_2| < \delta$. Since $\delta$ was arbitrary, we see that $f * g$ is uniformly continuous. □

b. If in addition $g$ is integrable, prove that $(f*g)(x) \to 0$ as $|x| \to \infty$.

*Proof.* Since both $f$ and $g$ are integrable, we have that $f*g$ is integrable. By a previous homework, we know that an uniformly continuous, integrable function tends to 0. Therefore, $(f*g)(x) \to 0$ as $|x| \to \infty$. □

Problem 4

Let $E \subset [0,1] \times [0,1]$ be a measurable set. Assume that $m(E_x) \leq \frac{1}{2}$ for almost every $x \in [0,1]$. Prove that $m(\{y \in [0,1] \mid m(E^y) = 1\}) \leq \frac{1}{2}$.

*Proof.* First, let $F$ denote the subset of $[0,1]$ where $m(E_x) \leq \frac{1}{2}$. We have that $m(F) = 1$. By a corollary of Tonelli's Theorem, we know that

$$\begin{aligned} m(E) &= \int_{[0,1]} m(E_x)dx \\ &= \int_F m(E_x)dx \\ &\leq \int_F \frac{1}{2}dx \\ &= m(F) \cdot \frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

Next, let $G$ denote the set $\{y \in [0,1] \mid m(E^y) = 1\}$. By the same corollary as before, we have

$$\begin{aligned} m(E) &= \int_{[0,1]} m(E^y)dy \\ &\geq \int_G m(E^y)dy \\ &= \int_G 1dy \\ &= m(G) \end{aligned}$$

From the previous observation, we know that $m(E) \leq \frac{1}{2}$. Therefore, $m(G) \leq \frac{1}{2}$, as desired. □

Problem 5

Let $f \in L^1(\mathbb{R})$ and define for $h > 0$

$$\phi_h(x) = \tfrac{1}{2h} \int_{x-h}^{x+h} f(t)dt$$

Prove that $\phi_h$ is integrable and $||\phi_h||_1 \leq ||f||_1$.

*Proof.* Define $F(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ so that

$$F(t,x) = f(t)\chi_A(t,x), \text{ where } A = \{(t,x) \mid x-h \leq t < x+h\}$$

Now, $A$ is the intersection of a closed half-plane and an open half-plane, so $A$ is measurable. Furthermore, $f \in L^1$, so $f$ is measurable. Taken together, we have that $F(t,x)$ is measurable. Now

$$\textstyle\int_{\mathbb{R}} F(t,x)dt = \int_{x-h}^{x+h} f(t)dt < \infty$$

and so

$$\begin{aligned} ||\phi_h||_1 &= \int_{\mathbb{R}} |\phi(x)|dx \\ &= \int_{\mathbb{R}} |\frac{1}{2h} \int_{x-h}^{x+h} f(t)dt|dx \\ &= \int_{\mathbb{R}} |\frac{1}{2h} \int_{\mathbb{R}} F(t,x)dt|dx \\ &\leq \frac{1}{2h}(\int_{\mathbb{R}} \int_{\mathbb{R}} |F(t,x)|dt)dx \\ &= \frac{1}{2h}(\int_{\mathbb{R}} \int_{\mathbb{R}} |F(t,x)|dx)dt && \text{(by Fubini's Theorem)} \\ &= \frac{1}{2h} \int_{\mathbb{R}} 2h|f(t)|dt && \text{(since } F(t,x) = 0 \text{ for } t \notin [x-h, x+h)\text{)} \\ &= \int_{\mathbb{R}} |f(t)|dt \\ &= ||f||_1 \end{aligned}$$

□

Problem 1

Consider the function on $\mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{1}{|x|(ln\frac{1}{|x|})^2} & \text{if } |x| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

a. Verify that $f$ is integrable.

*Proof.* We see that, for all $x \in \mathbb{R} \setminus \{0\}$, $f(x) \geq 0$ and $f$ is continuous (and so measurable). We compute the integral directly.

$$\begin{aligned}
\int_{\mathbb{R}} f(x)dx &= \int_{[\frac{-1}{2},\frac{1}{2}]} \frac{1}{|x|\left(ln\frac{1}{|x|}\right)^2}dx \\
&= 2\int_{[0,\frac{1}{2}]} \frac{1}{x\left(ln\frac{1}{x}\right)^2}dx \\
&= 2\lim_{h\to 0}\left[\frac{1}{ln\frac{1}{x}}\right]_h^{\frac{1}{2}} \\
&= \frac{2}{ln(2)} \\
&< \infty
\end{aligned}$$

□

b. Establish the inequality

$$f^*(x) \geq \frac{c}{|x|ln\frac{1}{|x|}}$$

for some $c > 0$ and all $|x| \leq \frac{1}{2}$ to conclude that the maximal function $f^*$ is not locally integrable.

*Proof.*

$$\begin{aligned}
f^*(x) &= \sup_B \frac{1}{m(B)}\int_B \frac{1}{|x|(ln\frac{1}{|x|})^2} \\
&\geq \frac{1}{2|x|}\int_{[-|x|,|x|]} \frac{1}{|x|(ln\frac{1}{|x|})^2} \\
&= \frac{2}{2|x|}\int_{[0,|x|]} \frac{1}{x(ln\frac{1}{x})^2} \\
&= \frac{1}{|x|ln\frac{1}{|x|}}
\end{aligned}$$

Now,

$$\begin{aligned}
\int_{[0,\frac{1}{2}]} |f^*(x)|dx &\geq \int_{[0,\frac{1}{2}]} \left|\frac{1}{|x|ln\frac{1}{|x|}}\right| dx && \text{(by above)} \\
&= \int_{[0,\frac{1}{2}]} \frac{1}{xln\frac{1}{x}}dx \\
&= \lim_{h\to 0}\left[ln\left(ln\left(\frac{1}{x}\right)\right)\right]_h^{\frac{1}{2}} \\
&= -\infty
\end{aligned}$$

Hence, $f^*$ is not locally integrable. □

Problem 2

Consider the function $F(x) = x^2 \sin\left(\frac{1}{x^2}\right)$, $x \neq 0$ with $F(0) = 0$. Show that $F'(x)$ exists for every $x$, but $F'$ is not integrable on $[-1, 1]$.

*Proof.* We have immediately that

$$F'(x) = 2\left(x\sin\left(\tfrac{1}{x^2}\right) - \tfrac{1}{x}\cos\left(\tfrac{1}{x^2}\right)\right)$$

which is finite everywhere except possibly at $x = 0$. We check this case separately using the definition.

$$\begin{aligned}
F'(0) &= \lim_{h\to 0}\frac{F(h)-F(0)}{h} \\
&= \lim_{h\to 0}\frac{h^2\sin\left(\frac{1}{h^2}\right)}{h} \\
&= \lim_{h\to 0} h\sin\left(\frac{1}{h^2}\right) \\
&= 0 && \left(\text{since } \sin\left(\frac{1}{h^2}\right) \text{ is bounded}\right)
\end{aligned}$$

To show that $F'(x)$ is not integrable on $[-1,1]$, it is sufficient to show that $\frac{1}{x}\cos\frac{1}{x^2}$ is not integrable over $[0,1]$. We accomplish this by approximating the area under the function by triangles.

$$\begin{aligned}
\int_{[0,1]}\frac{1}{x}\cos\frac{1}{x^2}dx &\geq \frac{1}{2}\sum_{k=1}^{\infty}\left(\left(\frac{\pi}{2}+(k+1)\pi\right)^{\frac{-1}{2}} - \left(\frac{\pi}{2}+k\pi\right)^{\frac{-1}{2}}\right)(k\pi)^{\frac{1}{2}} \\
&= \frac{1}{2}\sum_{k=1}^{\infty}\frac{\sqrt{k}}{\sqrt{(\frac{3}{2}+k)(\frac{1}{2}+k)}\cdot\left(\sqrt{\frac{3}{2}+k}+\sqrt{\frac{1}{2}+k}\right)}
\end{aligned}$$

Applying the limit comparison test (using $\frac{1}{k}$), we have

$$\begin{aligned}
\lim_{k\to\infty}\frac{k^{\frac{3}{2}}}{\sqrt{(\frac{3}{2}+k)(\frac{1}{2}+k)}\cdot\left(\sqrt{\frac{3}{2}+k}+\sqrt{\frac{1}{2}+k}\right)} &= \lim_{k\to\infty}\frac{1}{\sqrt{1+\frac{2}{k}+\frac{3}{4k^2}}\cdot\left(\sqrt{\frac{3}{2k}+1}+\sqrt{\frac{1}{2k}+1}\right)} \\
&= \frac{1}{2}
\end{aligned}$$

Since $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$ diverges, the limit comparison test implies that our original sum diverges, as well. Therefore, $\frac{1}{x}\cos\frac{1}{x^2}$ is not integrable over $[0,1]$, completing the proof. □

Problem 3

Suppose $F$ is of bounded variation and continuous. Prove that $F = F_1 - F_2$, where both $F_1$ and $F_2$ are monotonic and continuous.

*Proof.* Let $[a,b]$ with $a < b$ be any closed interval in $\mathbb{R}$. For $x \in [a,b]$, we have

$$\begin{aligned}
F(x) - F(a) &= P_a^x F - N_a^x F \\
F(x) &= (P_a^x F + F(a)) - N_a^x F.
\end{aligned}$$

Identifying $F_1$ with $P_a^x F + F(a)$ and $F_2$ with $N_a^x F$, it suffices to show that both $P_a^x F$ and $N_a^x F$ are continuous on $[a,b]$ (as they are obviously monotonic). Since

$$T_a^x F = P_a^x F - N_a^x F,$$

it further suffices to show that $T_a^x F$ is continuous on $[a,b]$. To see this, let $\tilde{x} \in [a,b]$ and let $\epsilon > 0$ be given. Since $F$ is uniformly continuous over the compact set $[a,b]$, we can find $\delta > 0$ small enough to ensure that $|F(x) - F(y)| \leq \frac{\epsilon}{3}$ whenever $|x-y| < \delta$ for all $x, y \in [a,b]$. Now, choose a partition $P$ of $[a,b]$ such that the distance between any two consecutive elements of the partition is less than $\delta$ and

$$T_a^b F < \sum_{k=1}^{N}|F(x_k) - F(x_{k-1})| + \frac{\epsilon}{3}$$

where the $x_k$ are the elements of $P$. Without loss of generality, we may assume that one of these elements, say $x_l$, is our $\tilde{x}$, since a refinement will only increase the precision of the estimate. Now, by restricting our view to the interval $[x_{l-1}, x_{l+1}]$, we have

$$\begin{aligned}
T_{x_{l-1}}^{x_{l+1}} F &< |F(x_l) - F(x_{l-1})| + |F(x_{l+1}) - F(x_l)| + \frac{\epsilon}{3} \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} && \text{(since } F \text{ continuous at } \tilde{x} \text{ and } x\text{'s are sufficiently close)} \\
&= \epsilon
\end{aligned}$$

Hence, the function $T_a^x$ is continuous on any interval $[a, b]$, thus proving the original claim. □

Problem 4

a. Let $F : [a, b] \to \mathbb{R}$ be a function of bounded variation and let $a < c < b$. Prove that $T_a^c F + T_c^b F = T_a^b F$. (Here, $T_a^b F$ denotes the total variation of $F$ on $[a, b]$.)

*Proof.* For any partition $P$, denote its elements by $x_k$.

$$\begin{aligned}
T_a^b F &= \sup_P \sum_{x_k \in [a,b]} |f(x_k) - f(x_{k-1})| \\
&= \sup_{P \cup \{c\}} \sum_{x_k \in [a,b]} |f(x_k) - f(x_{k-1})| && \text{(inclusion of the point } c \text{ can only refine } P\text{)} \\
&= \sup_{P \cup \{c\}} \left\{ \sum_{x_k \in [a,c]} |f(x_k) - f(x_{k-1})| + \sum_{x_k \in [c,b]} |f(x_k) - f(x_{k-1})| \right\} \\
&= \sup_{P \cup \{c\}} \sum_{x_k \in [a,c]} |f(x_k) - f(x_{k-1})| + \sup_{P \cup \{c\}} \sum_{x_k \in [c,b]} |f(x_k) - f(x_{k-1})| && \text{(since } [a,c] \text{ and } [c,b] \text{ are almost disjoint)} \\
&= T_a^c F + T_c^b F
\end{aligned}$$

□

b. Let $F$ be as in part (a). Prove that

$$\textstyle\int_a^b |F'(x)| dx \le T_a^b F.$$

*Proof.* Observe first

$$\begin{aligned}
T_x^{x+h} F &= \sup_P \sum_{x_k \in [x, x+h]} |F(x_k) - F(x_{k-1})| \\
&\ge |F(x+h) - F(x)| && \text{(this is the sum over a \textit{particular} partition of } [x, x+h]\text{)}
\end{aligned}$$

It follows

$$\begin{aligned}
|F'(x)| &= \lim_{h \to 0} \left| \frac{F(x+h) - F(x)}{h} \right| \\
&= \lim_{h \to 0} \frac{|F(x+h) - F(x)|}{h} \\
&\le \lim_{h \to 0} \frac{T_x^{x+h}}{h} && \text{(by above)} \\
&= \lim_{h \to 0} \frac{T_a^{x+h} - T_a^x}{h} && \text{(by part (a))} \\
&= (T_a^x F)'
\end{aligned}$$

Finally, we have

$$\begin{aligned}\int_a^b |F'(x)|dx &\leq \int_a^b (T_a^x F)' dx && \text{(by above)} \\ &= T_a^b F - T_a^a F \\ &= T_a^b F\end{aligned}$$

□

Problem 5

Let $a \leq b$ and define $F(0) = 0$, $F(x) = x^a \sin \frac{1}{x^b}$ for $0 < x \leq 1$. Prove that $F$ is not of bounded variation on $[0, 1]$.

*Proof.* Let $x_k = \left(\frac{\pi}{2} + k\pi\right)^{\frac{-1}{b}}$. Observe that, when $k$ is even, $\sin \frac{1}{x_k^b} = 1$, and when $k$ is odd, $\sin \frac{1}{x_k^b} = -1$. ow, for any finite sum of the $x_k$,

$$\begin{aligned}\sum_{k=1}^{N} |f(x_k) - f(x_{k-1})| &= \sum_{k=1}^{N} |(-1)^k (x_k^a + x_{k-1}^a)| \\ &= \sum_{k=1}^{N} (x_k^a + x_{k-1}^a) \\ &= x_N + x_0 + 2\sum_{k=1}^{N-1} x_k^a \\ &\geq \sum_{k=1}^{N-1} x_k^a \\ &= \pi^{\frac{-a}{b}} \sum_{k=1}^{N-1} \left(\frac{1}{2} + k\right)^{\frac{-a}{b}}\end{aligned}$$

which diverges as $N \to \infty$ (since $a \leq b$). Therefore, $F$ is not of bounded variation on $[0, 1]$. □

Problem 1

Let $F : [0, 1] \to \mathbb{R}$ such that $F'(x)$ exists almost everywhere and satisfies $F' \in L^1([0, 1])$. Assume $F$ is continuous at 0 and absolutely continuous on $[\epsilon, 1]$ for all $\epsilon > 0$. Prove that $F$ is absolutely continuous on $[0, 1]$ and thus of bounded variation on $[0, 1]$.

*Proof.* Since $F$ is absolutely continuous on $[\epsilon, 1]$, we have for any $\epsilon > 0$

$$F(x) = F(\epsilon) + \int_{\epsilon}^{x} F'(y)dy$$

for all $x \in [\epsilon, 1]$ (by the Second Fundamental Theorem of Calculus). Now, letting $\epsilon \to 0$

$$F(x) = F(0) + \lim_{\epsilon \to 0} \int_{\epsilon}^{x} F'(y)dy$$

for all $x \in [0, 1]$ (since $F$ is continuous at 0).

We claim that $\lim_{\epsilon \to 0} \int_{\epsilon}^{x} F'(y)dy = \int_0^x F'(y)dy$. For any $\epsilon > 0$, we have

$$\left|\int_0^x F'(y)dy - \int_{\epsilon}^{x} F'(y)dy\right| = \left|\int_0^{\epsilon} F'(y)dy\right|$$

which can be made arbitrarily small since $f \in L^1([0,1])$, thus proving the claim.

Therefore, we have that

$$F(x) = F(0) + \int_0^x F'(y)dy$$

for all $x \in [0,1]$, and so $F$ is absolutely continuous on $[0,1]$ (by the Second Fundamental Theorem of Calculus). As a consequence, we have that $F$ is of bounded variation on $[0,1]$. □

Problem 2

Let $a > b > 0$ and define $F(0) = 0$, $F(x) = x^a \sin\left(\frac{1}{x^b}\right)$ for $0 < x \leq 1$. Prove that $F$ is of bounded variation on $[0,1]$.

*Proof.* Observe that

$$F'(x) = ax^{a-1}\sin\left(\frac{1}{x^b}\right) - bx^{a-b-1}\cos\left(\frac{1}{x^b}\right)$$

which is defined on $(0,1]$. That is, $F'(x)$ exists almost everywhere on $[0,1]$.

We see that $F' \in L^1([0,1])$, since

$$\begin{aligned}
\int_0^1 |F'(x)|dx &= \int_0^1 \left|ax^{a-1}\sin\left(\frac{1}{x^b}\right) - bx^{a-b-1}\cos\left(\frac{1}{x^b}\right)\right| dx \\
&\leq \int_0^1 \left|ax^{a-1}\sin\left(\frac{1}{x^b}\right)\right| dx + \int_0^1 \left|bx^{a-b-1}\cos\left(\frac{1}{x^b}\right)\right| dx \\
&\leq \int_0^1 \left|ax^{a-1}\right| dx + \int_0^1 \left|bx^{a-b-1}\right| dx \\
&= \int_0^1 ax^{a-1}dx + \int_0^1 bx^{a-b-1}dx \\
&\leq 1 + \frac{b}{a-b} && \text{(since } a > b > 0\text{)} \\
&< \infty
\end{aligned}$$

Next, we claim that $F$ is continuous at 0, since

$$\begin{aligned}
\lim_{x\to 0} -x^a &\leq \lim_{x\to 0} F(x) &\leq \lim_{x\to 0} x^a \\
0 &\leq \lim_{x\to 0} F(x) &\leq 0
\end{aligned}$$

and $F(0) = 0$ by definition.

Now, since $F'(x)$ is integrable on $[\epsilon, 1]$ for all $\epsilon > 0$, we have that $F(x) = \int_\epsilon^x F'(y)dy$ is absolutely continuous on $[\epsilon, 1]$. By problem 1, we get further than $F(x)$ is absolutely continuous on $[0,1]$, and so of bounded variation on $[0,1]$. □

Problem 3

Let $f : [0,1] \to \mathbb{R}$. Prove that the following are equivalent.

1. $f$ is absolutely continuous, $f'(x) \in \{0,1\}$ almost everywhere, and $f(0) = 0$.
2. There exists a measurable set $A \subset [0,1]$ such that $f(x) = m(A \cap (0,x))$.

*Proof.* (1 ⇒ 2) Define $A$ to be the set $\{x \in [0,1] \mid f'(x) = 1\}$. Since $f$ is continuous, $f$ is measurable, so $f'$ is measurable, which in turn gives that $A$ is measurable. Now, since $f$ is absolutely continuous

$$\begin{aligned} f(x) &= f(0) + \int_0^x f'(y)dy \\ &= \int_0^x f'(y)dy \\ &= \int_0^x \chi_A(y)dy \\ &= \int_0^1 \chi_{A\cap(0,x)}dy \\ &= m(A \cap (0,x)) \end{aligned}$$

(2 ⇒ 1) We have immediately that

$$\begin{aligned} f(x) &= m(A \cap (0,x)) \\ &= \int_0^x \chi_A(y)dy \end{aligned}$$

and so $f$ is absolutely continuous (since $\chi_A \in L^1(\mathbb{R})$). By Lebesgue's Differentiation theorem, we have that $f'(x) = \chi_A(x)$ almost everywhere. Hence, $f'(x) \in \{0,1\}$ almost everywhere and $f(0) = 0$. □

Problem 4

Let $f_n$ be absolutely continuous on $[0,1]$ and let $f_n(0) = 0$. Assume that

$$\int_0^1 |f_n'(x) - f_m'(x)|dx \to 0$$

as $m, n \to \infty$. Prove that $f_n$ converges uniformly to a function $f$ on $[0,1]$ and that $f$ is absolutely continuous on $[0,1]$.

*Proof.* Since $L^1(\mathbb{R})$ is a Banach space, we know that the Cauchy sequence $\{f_n'\}$ converges to some function $g$ in norm. Let $f = \int_0^x g(y)dy$, which is absolutely continuous since $g \in L^1(\mathbb{R})$. We claim that $f$ satisfies the remaining criteria.

Observe that, since each $f_n$ is absolutely continuous and $f_n(0) = 0$,

$$\begin{aligned} f_n(x) &= f_n(0) + \int_0^x f_n'(y)dy \\ &= \int_0^x f_n'(y)dy \end{aligned}$$

Now

$$\begin{aligned} |f_n(x) - f(x)| &= \left| \int_0^x f_n'(y)dy - \int_0^x g(y)dy \right| \\ &= \left| \int_0^x f_n'(y) - g(y)dy \right| \\ &\leq \int_0^x |f_n'(y) - g(y)|dy \\ &\leq \int_0^1 |f_n'(y) - g(y)|dy \\ &\to 0 \end{aligned}$$

Hence, $f_n \to f$ pointwise. In fact, this convergence is uniform. Since the $f_n$ are absolutely continuous, they are of bounded variation, and so they are bounded. Hence, for each $n$, there is $M_n$ so that $|f_n - f| \leq M_n$ for all $x$. Since $f_n \to f$, $M_n \to 0$. So, given any $\epsilon > 0$, pick $N$ so that $M_n < \epsilon$ for all $n \geq N$. This gives $|f_n - f| < \epsilon$ for all $n \geq N$ and for all $x$. That is, $f_n \to f$ uniformly. □

Problem 5

Let $f : [a, b] \to [c, d]$ be an increasing absolutely continuous function and let $g : [c, d] \to \mathbb{R}$ be an absolutely continuous function. Prove that the composition $g \circ f : [a, b] \to \mathbb{R}$ is absolutely continuous.

*Proof.* Let $\epsilon > 0$ be given. Since $g$ is absolutely continuous, there exists $d > 0$ such that

$$\sum_{i=1}^{n} |g(d_i) - g(c_i)| < \epsilon$$

whenever $\{(c_i, d_i) \mid i = 1, \ldots, n\}$ are disjoint open intervals with $\sum_{i=1}^{n}(d_i - c_i) < \delta$. Similarly, since $f$ is absolutely continuous, there exists a $d' > 0$ such that

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \delta$$

whenever $\{(a_i, b_i) \mid i = 1, \ldots, m\}$ are disjoint open intervals with $\sum_{i=1}^{m}(b_i - a_i) < \delta'$. Hence, for any $\{(x_i, y_i) \mid i = 1, \ldots, l\}$ disjoint open intervals with $\sum_{i=1}^{l}(y_i - x_i) < \delta'$, have that that $\{(f(x_i), f(y_i)) \mid i = 1, \ldots, l\}$ are disjoint open intervals (since $f$ is increasing) with $\sum_{i=1}^{l}(f(y_i) - f(x_i)) < \delta$, and so $\sum_{i=1}^{l} |g(f(y_i)) - g(f(x_i))| < \epsilon$, as desired. □